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AN ARITHMETICAL THEORY OF CERTAIN NUMERICAL FUNCTIONS

by
ERIC TEMPLE BELL

SEATTLE, WASH.
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AN ARITHMETICAL THEORY OF CERTAIN NUMERICAL FUNCTIONS.

By E. T. Bell.

§ 0. Preliminary Considerations.

0.00. Apart from any evident utility as an economizer of thought and of calculation, there is, in the manifold interpretation of a system of postulates a wide philosophical significance.[1] The numerous instances of this multiplicity of meaning that have been devised in geometry, are common knowledge; in arithmetic the comparatively fewer examples, among which the Theory of Ideals of Dedekind is the classic, do not seem to be so generally appreciated, possibly because they lie slightly to one side of the main progress of analysis, although, as asserted by some,[2] arithmetic may be the proper foundation of all. The purpose of this paper is twofold; (i) To show, by several examples, that the postulates and processes of arithmetic admit of a multiplicity of interpretation, all examples to be simple and interconnected; (ii) To construct a self-contained *arithmetical* theory [cf. 0.01 (i)] of a large and important class of numerical functions, the theory to be so formed that the inter-relations of the functions considered [those in 5.23], shall be exhibited with a minimum of calculation, from either their symbolic or verbal definitions. Up to a certain point (i), (ii) are identical; beyond this, certain class-properties of the functions must be imagined in order to complete (i), and these newer aspects of the functions are relative only very remotely to the integers. The object in this part is to carry the developments sufficiently far to support several distinct interpretations of arithmetic [the simplest of all in this paper is in § 4], and the studies of congruences and forms [§ 11] for lack of space, have been deferred, although the methods in which they are to be approached are indicated. In all, sufficient material is provided for 32 distinct duals of arithmetic; in a narrower sense, it is shown [6.35], that an infinity of such may be evolved from a single mould. In the illustrations of (ii), given in §§ 7, 8, only those which may be briefly written, and not the most interesting, have been selected from a great quantity found by the methods of the paper. The more interesting applications arise from the consideration of (ideal) forms [§ 10]. Throughout, the insistence is upon methods rather than upon details of calculation which are elementary in all cases, and may be easily supplied if not indeed obvious.

[1] Cf. Cassius J. Keyser: *Concerning Multiple Interpretations of Postulate Systems and the "Existence" of Hyperspace.* [Journ. Phil., Psychology and Sci. Meth. (New York), Vol. IX, No. 10, 1913.]

[2] E. g., by Kronecker.

The properties of an integer may be regarded in two essentially distinct ways; either in relation to itself and its separate divisors, or in relation to all integers; these may be called the *static* and the *dynamic* aspect respectively, —relying on an obvious analogy. Of these, the static is the more easily investigated aspect; but, if (i) is to be accomplished, then clearly the dynamic must also be considered. In relation to (ii) the dynamic properties of *integers* are not taken account of, principally because of inherent difficulties that seem to place such considerations beyond the present reach of analysis, but also, because the theory of numerical functions in their dynamic aspects ceases to be arithmetical in the sense of [0.01], or in that well-expressed by E. Cahen: "In algebra, division is only exceptionally impossible, in arithmetic, only exceptionally possible."

Throughout, *, † refer respectively to definitions and theorems; thus * indicates that a section contains a definition essential to (i), †, that the section contains a theorem.

*0.01. (i) For precision in regard to 0.00 (i), an *arithmetical theory* may now be defined. It is emphasized that throughout, *addition, subtraction, multiplication and division, are used in their abstract meanings*, that is, as operations which obey the "ordinary abstract operational laws of algebra."[1] With respect to these operations,[2] a *number system*, N', that is also a *field*[1] is postulated. Let now a system of elements N be defined similarly to N' in all respects, *except that in N there are n identity elements*[2] *with respect to multiplication.* These n elements are the *units in N*. Elements of N that differ from one another only by unit factors are *equivalent*, and are considered *not distinct*.[3] Then, N is defined to be an *arithmetical field*, when and only when (1) and (2) hold:

(1) The assemblage of all elements in N is denumerable; and hence, if n is infinite, the units form a denumerable assemblage.

(2) Within N there is a denumerable assemblage, P, of elements, which are such that with respect to elements of P every element of N satisfies the *fundamental theorem of arithmetic*, viz., every element of N may be represented as a product of powers of distinct members of P in one way only. [Extensions will be dealt with as they arise naturally; e. g., in 4.64.]

(ii) The totality of relations between all members of an arithmetical field, the relations being with respect to the (abstract) fundamental laws of algebra, constitute an *arithmetical theory*.

(iii) Properties of the elements of N that are immediate consequences of

[1] E. H. Moore: Bull. Am. Math. Soc., vol. III (1893), p. 75. The order of the field is here infinite; if finite, a *finite* arithmetic is similarly, mutatis mutandis, defined; but finite arithmetics are not considered in this paper.

[2] For a very clear statement of what constitutes a number system, cf. O. Veblen and J. W. Young: *Projective Geometry*, vol. I (1910), p. 149. Cf. also J. König: ··· *Algebraischen Gröszen* (Leipzig, 1903), pp. 1–9.

[3] For the precise statement of the senses in which the units of this theory are respectively identical or distinct, cf. 3.29; 3.30; 3.31 and 6.03; 6.07; 6.09; 6.10.

the fundamental theorem of arithmetic, or which ultimately depend upon that theorem, are *arithmetical.*

(iv) If to (1), (2) be added further postulates, the whole being consistent, there results an *extended arithmetical theory.*

0.02. As arithmetic proper is only concerned with the properties of positive *integers,* so here, the main interest will be in the analogues of the integers. The number of units in this theory is infinite, but they preserve an important characteristic of unity, viz., their numerical values are identical. There is no concept of magnitude in the elements of this theory; hence, as the theorem that an integer is divisible by only a finite number of distinct integers, must here find an analogue, the finiteness of an element is defined otherwise than by its magnitude [cf. 6.13], but in a way equally applicable to the integers. Neither is there any concept of a natural order for the elements; an artificial order, having the properties in respect to the elements that the order of 1, 2, 3, $\cdots$, has to the integers, may be defined in connection with addition. Similarly for other concepts of arithmetic; these remarks indicate in what respects the theory of certain numerical functions will be shown to be isomorphic to arithmetic.

As addition, etc., are used often in the same context with abstract addition, etc., where necessary to distinguish between the kinds, that proper to this theory will be called *ideal addition,* etc. Moreover, in order to emphasize the abstract identity of certain of the operations used, with $+$, etc., the sign $+$ is sometimes used for a concept that arithmetically has the properties of $\times$ [cf. § 9]; this however is primarily done in accordance with usage.[1]

*0.03. A function, $f(x)$, is *numerical,* if $f(x)$ exists for every positive integral value of the *argument, x:* and moreover is such that $f(0) = 0$; $f(1) = 1$. By convention, a constant is a numerical function.

*0.04. A numerical function $f(x)$ [0.03] is *factorable,* if for every pair, n_1, n_2, of relatively prime values of the argument, $f(n_1)f(n_2) = f(n_1n_2)$.

*0.05. The *arithmetical definition* of a numerical function, $f(x)$, is the statement of the values assumed by the function according to the classes of values of the argument for which $f(x)$ exists. [For examples, cf. 2.01; 2.02; 2.04; 3.14.]

*0.06. If $f(x)$ is a numerical function, f is a *functional form:* and f is *abstracted* from $f(x)$.

0.07. The elements upon which 0.00 (i) is constructed are the functional forms of certain factorable numerical functions. The meaning of a functional form, which is, in effect, the arithmetical definition [0.05] of a numerical function, will be readily understood after the definition of ideal multiplication [3.07].

0.08. Sections 1, 2 are in a sense preliminary to the main part, which

[1] In the class-properties that are arithmetical, complete conformity with the rest of the theory may be attained on representing logical addition by $\times$; but as this is in violation of all precedents, it has not been done.

continues with 0.00 (i) in Section 3 "Ideal Multiplication." § 2 is primarily a source of future illustrations; and the purpose of § 1 is explained in 1.00.

§ 1. Simple Numbers.

1.00. There is a gain in conciseness in many subsequent theorems if an integer is regarded as a product of simple [1.01], instead of prime, numbers. Also, it will be shown that any theorem which depends ultimately upon the unique factorization theorem of arithmetic, is, according to respective resolutions into prime and simple numbers, susceptible of a dual interpretation [6.34; 6.35].

*1.01. A positive integer that is divisible by the square of no prime, is *simple*. Relatively prime simple numbers are *distinct*. Unity is simple, and other simple numbers are denoted by P_i $(i = 1, \cdots, \infty)$.

1.011. If n is resolved into its *prime* factors, this is also a resolution into *simple* factors; excluding this (unless n be simple, in which case the following resolution and the prime resolution coincide):

†1.02. Any number may be resolved into distinct simple factors in one way only. For, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the resolution of n into prime factors, the primes being so ordered that $\alpha_i \geqq \alpha_j$ when $i > j$; and for $s \leqq r$, let a_i, $(i = 1, \cdots, s)$ denote all the unequal α_i, $(i = 1, \cdots, r)$; also let $a_i = \alpha_1$, $a_s = \alpha_r$, and $a_1 < a_2 < \cdots < a_s$. Then, if P_i is the product of all those primes p_i which are such that $n/p_i^{a_i}$ is, but $n/p_i^{a_i+1}$ is not, an integer, the required *resolution of n into its simple factors* is obviously, $n = P_1^{a_1}P_2^{a_2}\cdots P_s^{a_s}$. The notations of 1.02, 1.03 prevail throughout the rest of this section, and the P_i, which are all distinct, are *the simple factors of n*. If $n = \pi_1^{\beta_1}\pi_2^{\beta_2}\cdots\pi_s^{\beta_s}$ represents either the resolution of n into its prime, or into its simple, factors, in the former case the π's are all distinct and relatively prime, in the latter, the β's are all distinct and the π's relatively prime, hence *distinct* [1.01].

*1.03. If $0 \leqq a_i' \leqq a_i$ $(i = 1, \cdots, s)$, and $n' = P_1^{a_1'}P_2^{a_2'}\cdots P_s^{a_s'}$, then n' is a *relative divisor* of n. If 1 is the only relative divisor of n, n', then n, n' are *relatively distinct;* in symbols, $Dv(n, n') = 1$. If $Dv(n_1', n_2') = 1$, then the *greatest common relative divisor* of nn_1', nn_2', is n; in symbols, if $n_1 = nn_1'$, $n_2 = nn_2'$, $Dv(n_1, n_2) = n$. Similarly the *lowest common relative multiple* of n_1, n_2, is defined by $Lm(n_1, n_2) = nn_1'n_2'$.

The number, $\varphi'(n)$, of integers $\not> n$ that are divisible by no simple factor except 1, of n, is the *relative totient* of n. The number, and the sum, of the relative divisors of n are denoted respectively by $\nu'(n)$ and $\sigma'(n)$. In order to indicate that n is being considered as a product of simple rather than of prime factors, n is replaced by n'; numerically, $n = n'$. The function $\mu'(n)$, or $\mu'(n')$ has the value 0 if n' is divisible by the square of any simple number, and otherwise, is $+ 1$ or $- 1$ according as n' is the product of an even or of an odd number of simple numbers.

†1.04. The following are immediate consequences of the definitions;

$$\text{(i)}\ \nu'(n') = \prod_{i=1}^{s}(a_i + 1);\ \text{(ii)}\ \sigma'(n') = \prod_{i=1}^{s}(1 - P_i^{a_i+1})/(1 - P_i);$$

(iii) $Dv(n_1', n_2')\cdot Lm(n_1', n_2') = n_1'n_2'$; (iv) $\varphi'(n') = n'\prod_{i=1}^{s}(1 - 1/P_i)$.

The last may be deduced similarly to the ordinary totient, $\varphi(n)$, from the principle of cross-classification of classes,[1] noting that in their totality the P_i are relatively prime [1.02].

†1.05. If $Dv(n_1', n_2') = 1$, and if $\psi'(n')$ is any one of the functions $\nu'(n')$, $\sigma'(n')$, $\varphi'(n')$, then clearly, $\psi'(n_1'n_2') = \psi'(n_1')\psi'(n_2')$. The like is not, in general, true if $Dv(n_1', n_2') > 1$.

†1.06. As in the corresponding theorem for $\varphi(n)$, it may be shown that $\Sigma\varphi'(d') = n'$, the summation referring to all relative divisors d' of n'.

†1.07. If $\psi'(n')$ is *any* function such that $\psi'(n_1'n_2') = \psi'(n_1')\psi'(n_2')$ when $Dv(n_1', n_2') = 1$, and if the summations refer to all relative divisors d' of n', then if $\Psi'(n') = \Sigma\psi'(d')$, is

$$\psi'(n') = \Sigma\mu'(d')\Psi'(n'/d').$$

The proof is similar to that usually given when the resolution of n (or n') is by prime, instead of by simple, factors. [Or, cf. § 8.]

†1.08. All of the theorems in 1.04 to 1.07 are special cases of a general result that now may be readily inferred; however, for the sake of uniformity of treatment, a set of ideal elements, or *l-numbers* is introduced. Any particular result of the above kind may be proved at once from first principles, but the systematic finding of such is better done otherwise.

*1.09. The *l-numbers* 1, l_i $(i = 1, 2, \cdots, \infty)$ are such that $l_i \neq l_j$ when $i \neq j$, and are placed in (1, 1) correspondence with the natural primes, p_i, so that l_i, p_i are correspondents. Hence, to any integer $n \equiv p_1^{a_1}p_2^{a_2}, \cdots$, corresponds a definite symbol, $l_1^{a_1}l_2^{a_2}\cdots$. Recasting now the definitions in 1.01, 1.02, replacing therein any prime p_i by l_i, and defining any $l_1^{a_1}l_2^{a_2}\cdots$ as an *l-symbol*, the terms *simple l-symbol* and *resolution of an l-symbol into its simple factors*, may be taken as defined; also, *relatively distinct l-symbols* are now defined. The l-symbol corresponding to n' is denoted by $L(n')$, and $\{L(n')\}^r \equiv L^r(n')$. Thus a new set of *L-numbers* is defined, whose characteristic properties, and the only ones that are relevant for the present purpose, are *by definition; the product of any number of L-numbers is 1 or 0 according as the L-numbers are or are not relatively distinct in their totality. Multiplication of L-numbers is commutative and associative: also, if m is any integer,* $m \times L(n') = L(n') \times m$.

†1.10. Considering now the *formal* expansion of each factor, and subsequent distribution of the product, it follows from the definitions in 1.09 that $S(s) \equiv \prod_{i=1}^{\infty}(1 - L(P_i)/P_i^s)^{-1}$ is equivalent formally to $\sum_{n'=1}^{\infty} 1/n'^s$. For, any term in the distributed product is of the form $L^{a_i}(P_i)L^{a_j}(P_j)\cdots L^{a_k}(P_k)/$

[1] Cf. H. J. S. Smith: *On the History of the Researches of Mathematicians on the Subject of the Series of Prime Numbers.* Proc. Ashmolean Soc., III, 128–131; or Coll. Papers, I, p. 36. Or, more briefly as in § 8.

$(P_i^{\alpha_i}P_j^{\alpha_j}\cdots P_k^{\alpha_k})^s$; whose numerator is 1 when and only when the denominator is the resolution into simple factors of some number, and otherwise is 0; also, every possible product of powers of simple numbers, and hence every number, occurs in the denominators. Hence if s is so chosen that the series converges, the infinite product and series may be equated.

1.11. Most of the simple-number analogues of elementary arithmetical theorems are readily suggested and easily proved; thus if n_1', n_2' are relatively simple, and if n_1', n_2' is relatively divisible by n', then so also is only one of n_1', n_2'. But sometimes the analogy is not so close; thus, if P is simple, $P!$ and P are relatively distinct. It suffices to show that if $P = p_1p_2\cdots p_r$, the exponents of the highest powers of p_1, p_2 that divide $P!$ are unequal. Let $p_1 < p_2$, and let $[m/n]$ denote the integral part of m/n: the exponents in question are $P/p_i + [P/p_i^2] + [P/p_i^3] + \cdots$; $(i = 1, 2)$. Since $p_1 < p_2$, $P/p_1 > P/p_2$; whence, for $\alpha \geqq 2$, $[P/p_1^\alpha] \geqq [P/p_2^\alpha]$; hence the exponents are unequal. Usually it is sufficient to replace *number*, by *simple number*, *prime*, by *simple*, but not always; e. g., $\varphi'(n')$ is not the number of integers $\not> n$ and *relatively distinct* to n. The exact meaning of *relatively prime to* n must be transformed into *not divisible by any simple factor of* n'. With such changes the true analogue of any theorem is found without much difficulty; numerous examples will be found later [cf. 6.34].

§ 2. The Function $\Psi(n;\ a, b, c, l)$.

2.00. The majority of factorable numerical functions in current use, and most of those used later for purposes of illustration, are specializations of a single simple function, Ψ, [2.01]. In this Ψ, $l = +1$ or -1, a, b are positive integral constants, n is a variable integer, and c is finite, otherwise arbitrary. This Ψ includes the numerous functions considered by Liouville [§ 7], and, in its general form, wherein l is arbitrary, ∞^4 in all. As defined in 2.01, Ψ is the simplest case of a more general function [7.02 (ii)], whose verbal definition is so prolix, that it is best deferred until several new concepts shall have been introduced [§§ 3, 5]. All of these in turn constitute the simplest possible example of a *class* of numerical functions,[1] which is the first in an infinite assemblage of such classes.

*2.01. Two auxiliary functions are first defined:

(i) $\lambda(n)$; the *multiplicity* of n; that is, the *total* number of primes which divide n.

(ii) $\gamma(n)$; the *manifoldness* of n; that is, the number of *distinct* primes which divide n. [This term is due to Sylvester.] Then $\Psi(n;\ a, b, c, l)$, for $l = \pm 1$ is defined by (iii) and (iv):

(iii) $\Psi(n;\ a, b, c, 1) = 0$ if n is not the perfect bth power of a simple number [1.01]; and in the contrary case, $= c^{\gamma(n)}n^{a/b}$.

(iv) $\Psi(n;\ a, b, c, -1) = 0$ if n is not the perfect bth power of an integer; and in the contrary case, $= (-c)^{\lambda(n^{1/b})}n^{a/b}$.

[1] Considered in 3.14; 5.24 (iii). The $\gamma(n)$ in 2.01 (ii) will not be confused with $\gamma(H)$ of § 5.

In both (iii), (iv), $n^{a/b}$ signifies the *arithmetical* bth root of n^a. Clearly, if n is simple, $\gamma(n) \equiv \lambda(n)$.

*2.02. As a first specialization of Ψ, the functions on the left of the respective identities are defined on referring to 2.01 [cf. also 2.05]:

$$\left.\begin{array}{l}\psi(n;\ a,\ b) \equiv \Psi(n;\ a,\ b,\ -1,\ \ 1)\\ \psi'(n;\ a,\ b) \equiv \Psi(n;\ a,\ b,\ -1,\ -1)\end{array}\right\};\quad \left.\begin{array}{l}\chi(n;\ a,\ b) \equiv \Psi(n;\ a,\ b,\ 1,\ \ 1)\\ \chi'(n;\ a,\ b) \equiv \Psi'(n;\ a,\ b,\ 1,\ -1)\end{array}\right\};$$

$$\left.\begin{array}{l}\zeta(n;\ a,\ b) \equiv \Psi(n;\ 0,\ b,\ a,\ \ 1)\\ \zeta'(n;\ a,\ b) \equiv \Psi(n;\ 0,\ b,\ a,\ -1)\end{array}\right\}.$$

Each pair is evidently a particularization of the pair $\Psi(n;\ a,\ b,\ c,\ \pm 1)$, whose fundamental property may be stated, although not proved until 7.04:

†2.03. The summation extending to all divisors D, of N, or to all relative divisors D, of N, according as N is regarded as a product of prime or of simple numbers, the value of

$$\Sigma\Psi(D;\ a,\ b,\ c,\ l)\Psi(N/D;\ a,\ b,\ c,\ -l)$$

is 1 or 0 according as $N = 1$ or $N > 1$.

Putting $l = 1$, the meaning of the theorem is seen for the functions in 2.02.

*†2.04. Of the six functions in 2.02, sixteen sub-cases are of such frequent occurrence in arithmetic, that they have received special notations, now given; some of the symbols are due to LIOUVILLE[1] and other writers on the subject, but no attempt at conformity with these in all has been made, as the subsequent point of view is distinct. Comparing with 2.02, 2.041 for verifications of the implied theorems, the sixteen are [cf. also § 7]:

$$\begin{array}{rll}
\text{(i)} & \psi(n;\ 1,\ 2) & \equiv \mu(\sqrt{n})k_2(n)u_1(\sqrt{n}).\\
\text{(ii)} & \psi(n;\ r,\ 1) & \equiv \mu(n)u_r(n).\\
\text{(iii)} & \psi'(n;\ 1,\ 1) & \equiv u_1(n).\\
\text{(iv)} & \psi'(n;\ 1,\ 2) & \equiv k_2(n)u_1(\sqrt{n}).\\
\text{(v)} & \psi'(n;\ r,\ 1) & \equiv u_r(n).\\
\text{(vi)} & \chi(n;\ 1,\ 1) & \equiv \{\mu(n)\}^2u_1(n).\\
\text{(vii)} & \chi'(n;\ 1,\ 1) & \equiv \varpi(n)u_1(n).\\
\text{(viii)} & \chi'(n;\ r,\ 1) & \equiv \varpi(n)u_r(n).\\
\text{(ix)} & \zeta(n;\ r-1,\ 1) & \equiv \zeta_{r-1}(n) \equiv \{\mu(n)\}^2(r-1)^{\nu(n)}.\\
\text{(x)} & \zeta(n;\ 2r-1,\ 1) & \equiv \zeta_{2r-1}(n) \equiv \{\mu(n)\}^2(2r-1)^{\nu(n)}.\\
\text{(xi)} & \zeta(n;\ 1,\ 1) & \equiv \{\mu(n)\}^2.\\
\text{(xii)} & \zeta(n;\ 2,\ 1) & \equiv \{\mu(n)\}^2\nu(n).\\
\text{(xiii)} & \zeta(n;\ -1,\ 1) & \equiv \mu(n).\\
\text{(xiv)} & \zeta(n;\ 2^r-1,\ 1) & \equiv \{\mu(n)\}^2(2^r-1)^{\nu(n)}.\\
\text{(xv)} & \zeta'(n;\ 1,\ 1) & \equiv \varpi(n).\\
\text{(xvi)} & \zeta'(n;\ -1,\ 1) & \equiv u_0(n).
\end{array}$$

*2.041. The definitions of the various symbols on the right of the sixteen

[1] References in § 7.

identities in 2.04, are, for n a positive integer, r a positive integer unless the contrary is expressly stated:

(i) $\mu(n)$ is the function of MÖBIUS (sometimes, of MERTENS), and vanishes if n is not simple, and otherwise is $+1$ or -1 according as the manifoldness of n is even or odd. $\mu_{1/a}(n) \equiv \mu(\sqrt[a]{n})$, which exists only when n is a perfect ath power; $\sqrt[a]{n} \equiv +\sqrt[a]{n}$, arithmetical root.

(ii) $k_r(n) = 1$ or 0 according as n is or is not a perfect rth power.

(iii) $u_r(n) = n^r$; $u_1(\sqrt{n})$ may be written $u_{1/2}(n)$, or $u^{1/2}(n)$; similarly for higher roots.

(iv) $\varpi(n) = +1$ or -1 according as the multiplicity of n is even or odd; viz., $\varpi(n) = (-1)^{\lambda(n)}$.

(v) $\nu(n) =$ the number of divisors of n. In addition to these, for completeness;

(vi) $\varphi(n) =$ the *totient* of n: viz., the number of integers $\not>$, and prime to, n.

(vii) $\sigma(n) =$ the sum of the divisors of n.

(viii) $\theta(n) =$ the *total* number of decompositions of n into a pair of relatively prime factors; in this, if $n = n_1n_2$ is one such decomposition, n_2n_1 is to be counted as another, distinct from the former.

(ix) $\{D_r(n)\}^r =$ the greatest perfect rth power that divides n; written $D_r^{(r)}(n)$. Among the divisors of n are included *always*, 1 and n; also, if any of these functions is undefined (by its nature), or is ambiguous, for $n = 1$, by convention, the value is taken to be unity.

† 2.05. Comparing the definitions of the functions in 2.02 with that of the Ψ-function in 2.01, it is easy to verify that:

$$
\begin{aligned}
&\text{(i)} & \psi\,(n;\ a,\ b) &= k_b(n)\cdot\mu_{1/b}(n)\cdot u_{a/b}(n);\\
&\text{(ii)} & \psi'(n;\ a,\ b) &= k_b(n)\cdot u_{a/b}(n).\\
&\text{(iii)} & \chi\,(n;\ a,\ b) &= k_b(n)\cdot\{\mu_{1/b}(n)\}^2\cdot u_{a/b}(n);\\
&\text{(iv)} & \chi'(n;\ a,\ b) &= (-1)^{\lambda(n^{1/b})}\cdot k_b(n)\cdot u_{a/b}(n).\\
&\text{(v)} & \zeta\,(n;\ a,\ b) &= \{\mu_{1/b}(n)\}^2\cdot k_b(n)\cdot a^{\gamma(n)};\\
&\text{(vi)} & \zeta'(n;\ a,\ b) &= (-1)^{\lambda(n^{1/b})}\cdot k_b(n)\cdot a^{\lambda(n^{1/b})}.
\end{aligned}
$$

The verbal equivalents are:

(i) $\psi(n;\ a,\ b) = 0$ if n is not the bth power of a simple number, and, in the contrary case, $= \pm n^{a/b}$ according as the multiplicity (or, what is here equivalent, the manifoldness), of $n^{1/b}$ is even or odd.

(ii) $\psi'(n;\ a,\ b) = 0$ or $n^{a/b}$ according as n is not or is a perfect bth power.

(iii) $\chi(n;\ a,\ b) = 0$ or $n^{a/b}$ according as n is not or is a perfect bth power of a simple number.

(iv) $\chi'(n;\ a,\ b) = 0$ if n is not a perfect bth power, and in the contrary case, $= \pm n^{a/b}$ according as the multiplicity of $n^{1/b}$ is even or odd.

(v) $\zeta(n;\ a,\ b) = 0$ or $a^{\gamma(n)}$ according as n is not or is a perfect bth power of a simple number.

(vi) $\zeta'(n;\ a,\ b) = 0$ if n is not a perfect bth power, and in the contrary case is $\pm a^{\lambda(n^{1/b})}$ according as the multiplicity of $n^{1/b}$ is even or odd.

2.06. In the set of sixteen [2.04], (ix), (x), (xiv) appear but slightly different from each other; when their inter-relations with the others come to be examined, it will be found that they are essentially distinct. Anticipating, if either $a = b = 1$, or if otherwise, a, b are unequal, it will be shown that the six functions of 2.02, and hence the sixteen of 2.04, when *considered as functions are ideal primes*, although several, e. g., (i), (ii), (iv) of 2.04 apparently contradict any notion of primeness. Also, $\nu(n)$, $\sigma(n)$, $\varphi(n)$, $\theta(n)$, and many others will be exhibited as ideal products of powers of the sixteen primes. It is the prime property that makes these specializations of Ψ important for illustrations, etc.

§ 3. Definition of Ideal Multiplication; etc.

3.00. The concept of ideal multiplication may be briefly illustrated by the first (historically) example in which it is implicit. Denoting by $\varphi(n)$ the totient of n (viz., the number of integers not greater than, and prime to, n), and by $u_0(n)$ a numerical function of n which has the value unity for all values of the argument n, and by $u_1(n)$, a numerical function which has the value n for all values of the argument n, there is the well-known theorem $\sum_{(n)}\varphi(d) = n$; or what is equivalent,

$$\text{(i)} \qquad \sum_{(n)} \varphi(d)u_0\left(\frac{n}{d}\right) = u_1(n);$$

the notation $\sum_{(n)}$, etc.; meaning (as customary) that the summation is extended over all divisors d, including 1 and n, of n.

Modifying the notation, the *equality* (or *identity*) (i) may be rewritten, $\varphi u_0 \sim u_1$; and is to be read, "the functional form u_1 is *equivalent*, $\sim$, to the symbolic, or *ideal* product of the functional forms φ and u_0"; or, φ and u_0 constitute a pair of *ideal divisors* of u_1"; or, "u_1 is *divisible by* φ and u_0 *completely*"; and $\varphi u_0 \sim u_1$ is an *equivalence of functional forms*. Throughout, n is any positive integer.

*3.01. If ψ, ψ', ψ'' are functional forms [0.06], which are such that for all positive integral values of the argument n,

$$\text{(i)} \qquad \sum_{(n)} \psi'(d)\psi''\left(\frac{n}{d}\right) = \psi(n),$$

the Σ notation being that of 1.00; the equality (i) being rewritten in the form $\psi'\psi'' \sim \psi$, each of the functional forms ψ', ψ'' is called an *ideal divisor of the functional form* ψ, and ψ is the *ideal product* (with respect to *ideal multiplication*) of ψ', ψ''; also, ψ is *completely divisible* by ψ' and ψ''.

Henceforth, if f_1, f_2 are any functional forms, f_1f_2 shall signify an ideal product as defined; viz., f_1f_2 shall mean $\sum_{(n)} f_1(d)f_2\left(\frac{n}{d}\right)$, *wherein the argument n is general.* The sign "$\sim$" is read, "is equivalent to," and expresses the precise relation above defined; $\psi'\psi'' \sim \psi$ is an *equivalence.*

†3.02. As a relation, equivalence is symmetrical and transitive; viz., (i) if $\psi'\psi'' \sim \psi$, then $\psi \sim \psi'\psi''$; (ii) if $\psi \sim \psi'$ and $\psi' \sim \psi''$, then $\psi \sim \psi''$. (iii) $\psi'\psi'' \sim \psi''\psi'$; that is, ideal multiplication is commutative. These are obvious consequences of 3.01. The concepts of 3.01 are now made general[1] so as to apply to any number of functional forms, $\psi, \psi_1, \psi_2, \cdots$.

*3.03. Denote by d_1', d_2 a pair of conjugate divisors of n, so that $n = d_1'd_2$; resolving similarly, let $d_2 = d_2'd_3$; $\cdots$ etc.; whence, finally,

$$\text{(i)} \qquad \begin{cases} n = d_1'd_2, \\ d_2 = d_2'd_3, \\ d_3 = d_3'd_4, \\ \cdots\cdots\cdots\cdots \\ d_{r-2} = d_{r-2}'d_{r-1}, \\ d_{r-1} = d_{r-1}'d_r'. \end{cases}$$

Let $\psi_1(n_1), \psi_2(n_2), \cdots, \psi_r(n_r)$ denote numerical functions [0.03], and consider the r-fold summation, $S_r(n)$, where

$$\text{(ii)} \qquad S_r(n) \equiv \sum_{(n)}\sum_{(d_2)}\sum_{(d_3)}\cdots\sum_{(d_{r-2})}\sum_{(d_{r-1})}\psi_1(d_1')\psi_2(d_2')\psi_3(d_3')\cdots\psi_r(d_r');$$

the summations on the right being over all values of the product $\psi_1(d_1')\psi_2(d_2')\cdots\psi_r(d_r')$ which are formed by supplying the arguments d_i' to $\psi_i(n_i)$ in $\psi_1(n_1)\psi_2(n_2)\cdots\psi_r(n_r)$; $(i = 1, \cdots, r)$ from *all possible solutions* of the set (i). Then;

†3.04. $S_r(n)$ is a numerical function.

†3.05. If $\psi_i(n)$ $(i = 1, \cdots, r)$ are factorable numerical functions [0.04], then $S_r(n)$ is a factorable numerical function.

3.06. The proofs of 3.04, 3.05, are immediately evident from the definitions.

*3.07. Considering[1] $S_r(n)$, or for uniformity $\psi(n)$, $[\equiv S_r(n)]$ as a numerical function *of the general argument n,* the identity in (ii) will be written in the symbolic (or ideal) form,

$$\text{(i)} \qquad \psi \sim \psi_1\psi_2\cdots\psi_r.$$

Conversely, if a functional form ψ be given, and it is possible to determine

[1] Throughout the entire paper, it will be well to remember a saying of EDOUARD LUCAS [*Théorie des Nombres,* Paris (1891), p. 205]: "The symbolic calculus must be considered as a rapid method for the writing of formulae in a series of theoretical deductions; but, whenever it is a question of determining the values of the numbers furnished by this calculus, it is indispensable to replace the symbolic formula by the ordinary development. . . . It is then, in a certain measure, a shorthand of the formulae of arithmetic and algebra for the development of new theories." The symbolic calculus developed here is, of course, distinct from that of LUCAS, but the latter may be applied to the former, since, as will be shown, the theory of numerical functions is isomorphic to both algebra and arithmetic,—thus giving rise to new functions or new properties of the present functions; but this does not properly belong to this part. The remark of LUCAS is especially pertinent to 0.00 (ii).

functional forms $\psi_1, \psi_2 \cdots, \psi_r$ in such a way that 3.03 (ii) is an identity for all values of n, then this fact is expressed by 3.07 (i), which is to be read: *the functional form ψ is equivalent to the ideal product of the functional forms $\psi_1, \psi_2, \cdots, \psi_r$*; or, *the functional form ψ is divisible by the functional forms $\psi_1, \psi_2, \cdots, \psi_r$, completely;* also, each of ψ_i $(i = 1, \cdots, r)$ is an *ideal divisor* of ψ, and the operation of forming the product (ideal) of $\psi_1, \psi_2, \cdots, \psi_r$ is *ideal multiplication.*

†3.08. Ideal multiplication is associative. It will be sufficient to show that $(\psi_1\psi_2)\psi_3 \sim \psi_1(\psi_2\psi_3)$, the left of this equivalence denoting the ideal product of ψ_3 and the ideal product of ψ_1, ψ_2, the right, denoting the ideal product of ψ_1 and the ideal product of ψ_2, ψ_3; (for, by 3.02 (iii) ideal multiplication is commutative). The theorem follows at once from 3.03 (i), (ii), remarking that the totalities of all solutions of $n = d_1d_2$, $d_1 = d_1'd_1''$, and $n = \delta_1\delta_2$, $\delta_2 = \delta_2'\delta_2''$, for any particular value of n, are identical (except for order), the totalities being respectively all values of d_1', d_1'', d_2 and $\delta_1, \delta_2', \delta_2''$ that satisfy the equations, $n = d_1'd_1''d_2$; $n = \delta_1\delta_2'\delta_2''$.

3.09. Within the entire class of numerical functions, it is not difficult to show [cf. § 5] that, in the sense of 3.07, *any functional form ψ is divisible by any other functional form ψ'*; moreover, it may be shown that *if ψ, ψ' are given, ψ'' may be determined in an infinity of ways so that $\psi \sim \psi'\psi''$; ψ, ψ', ψ'' being numerical functions.* Hence if an arithmetical theory [0.01] of the numerical functions is to be constructed, their definition must be modified. It is the purpose of the sections following to select from the entire class of functions defined in 0.03 a denumerably infinite sub-class for which an arithmetical theory exists. Henceforth, the juxtaposition of functional forms shall signify the ideal product of the forms, and the qualification *ideal* may be dropped in referring to multiplication.

*3.10. If $\psi(n)$ is a numerical function which vanishes for all values of n greater than unity, then $\psi(n)$ is a *unit function,* and ψ is a *unit functional form;* briefly, ψ is a *unit.* [For examples, cf. 2.03; 7.04; 7.09.] Units will be denoted by $\epsilon, \epsilon', \cdots, \epsilon_1, \epsilon_2, \cdots$, etc.

*3.11. If ψ is the product of several functional forms, of which at least two are distinct from units, ψ is *composite.* [Example 3.00; u_1 is composite; but, cf. 3.35.]

*3.12. If for all positive integral values of n,[1] $\psi(n) = \psi'(n)\psi''(n)$ (*algebraic* product), then ψ is a *compound function;* and $\psi \sim |\, \psi'\psi'' \,|$, this being the definition of the symbol $|\, \psi'\psi'' \,|$: viz., $|\, \psi'\psi'' \,|$ is a *compound* functional form. [Examples in 2.04 (i), (ii), (iv); (vi) to (xii); (xiv).]

3.13. In general, $\psi'\psi'' \sim |\, \psi'\psi'' \,|$ is obviously false. The distinction between compound and composite ψ_i is very important for the sequel. Clearly, the ψ_i might have been classified according to their factors when considered as compound functions; a slight consideration is sufficient to convince that

[1] Henceforth n shall always signify a positive integer; n is the symbol of *any* integer, not of a *particular* integer. Also, $\psi, \psi', \cdots$ etc.; $\psi_1, \cdots$ are functional forms as defined in 0.06.

no appreciable progress can be made in regard to either (i) or (ii) of 0.00 in this direction, and that such a classification is unnatural. The primitives defined in 3.14 are fundamental in the theory of the ψ_i, also for all that follows.

*3.14. Let $(t) \equiv t_1, t_2, \cdots, t_r$ denote a set of positive non-zero integers, no two of which are equal, and $(t') \equiv t_1', t_2', \cdots, t_r'$ any permutation of (t). Also, let $(h) \equiv h_1, h_2, \cdots, h_r$ denote a set of functional forms abstracted from factorable numerical functions $h_1(x), h_2(x), \cdots, h_r(x)$ respectively [0.06; 0.04], no one of which is a unit [3.10], but any one of which may reduce to a numerical constant, and let a (1, 1) correspondence be established between (t) and (h) in such a way that t_i, h_i are correspondents, and denote by $(h') \equiv h_1', h_2', \cdots, h_r'$ that permutation of (h) in which h_i' corresponds to t_i' $(i = 1, \cdots, r)$. Then, if $P_1^{\tau_1}P_2^{\tau_2} \cdots P_s^{\tau_s}$ is the resolution of n into its simple factors [1.02], a function, $H(n)$, of n may be defined unambiguously by the properties (i), (ii):

(i) $H(n)$ vanishes when any τ_i $(i = 1, \cdots, s)$ is not a member of (t); hence in particular, $H(n) = 0$ if $s > r$.

(ii) If every τ_i is a member of (t), and if $\tau_i = t_i'$ $(i = 1, \cdots, s)$, then $H(n) = h_1'(P_1)h_2'(P_2) \cdots h_s'(P_s)$. [Cf. also 5.14.]

The so-defined $H(n)$ is a *primitive function*, or simply, a *primitive*.

†3.15. A primitive is a factorable numerical function. This follows at once from the definitions in 3.14; 0.04

3.16. In order to specify completely a primitive, it is necessary to give both the sets of integers and functions and the correspondence between them, from which the primitive is derived. In 3.17 to 3.19, the notation is that of 3.14. The definitions in 3.18, 3.19 are again fundamental.

*3.17. H is a *primitive form.*

*3.18. Let $(t''), \equiv t_1'', t_2'', \cdots, t_r''$ denote that permutation of (t) which is such that $t_r'' > t_{r-1}'' > \cdots > t_2'' > t_1''$, and let h_i'' denote the correspondent of t_i'' $(i = 1, \cdots, r)$. Then (t'') is the *index of the primitive form* H, written, $I(H)$; and (h'') is the base of H, written $B(H)$; $\equiv h_1'', h_2'', \cdots, h_r''$.

*3.19. The index $I(H)$ and the base $B(H)$ [3.18] taken together, and written in the form $\begin{bmatrix} I(H) \\ B(H) \end{bmatrix}$, constitute the *characteristic* of H, written $K(H)$ for brevity.

†3.20. If $K(H)$ is given, where H is a primitive form, the primitive $H(n)$ is uniquely defined; obviously.

†3.21. (i) $\Psi(n;\ a, b, c, 1)$ is a primitive function [2.01].

(ii) If $h(x) \equiv cx^a$, then $K(\Psi) = \begin{bmatrix} b \\ h \end{bmatrix}$; [2.01; 3.19]; Ψ being abstracted from $\Psi(n;\ a, b, c, 1)$.

(iii) $\psi(n;\ a, b)$; $\chi(n;\ a, b)$; $\zeta(n;\ a, b)$ are primitives [2.02].

(iv) The right-hand members of 2.04 (i), (ii), (vi); (ix) to (xiv), are primitives. Of these, (i), (ii) may be verified from the definitions; (iii) follows from (i), and (iv) from (iii), referring to 2.02, 2.04. As all these are mere illustrations, and are not used in the deduction of any subsequent theorems,

until § 8, the verification of (i), (ii) may be omitted if desired, until §§ 5; 7, when both are proved instantaneously.

*3.22. In the definition of primitives [3.14], there is an apparent narrowing of the development by the restriction that the $h_i(x)$ shall be factorable. This is now removed as follows: *if $h(x)$ is not factorable, and if $P \equiv p_i p_j \cdots p_k$ is a simple number, then $h(P)$ shall mean $h(p_i)h(p_j) \cdots h(p_k)$.* Clearly, in this case, $h(P)$ is purely symbolic. An important restriction is now imposed upon the primitives.

*3.23. If the functions $h_i(x)$ used in defining a primitive form H [3.14; 3.17], are polynomials in x with positive, negative or zero integers as coefficients, the case in which every coefficient is zero being excluded by 3.14, then $H(n)$ is an *algebraic primitive*, and H is an *algebraic primitive form;* provided the *degrees* of the polynomials be all *finite* (if this is not included in the definition as usually given, of a polynomial).

*3.24. Referring to 3.14, r is the *extent* of $H(n)$ and of H. The extent is of subsequent importance in that the case in which r is infinite (essentially), gives an analogue to the entire transcendental functions. [Cf. 5.30.]

*3.25. Let H_i $(i = 1, \cdots, r)$ denote algebraic primitive forms. Then the meaning of $H \sim H_1H_2\cdots H_r$ is known from 3.07; and if $H_1, H_2, \cdots, H_r$ are identical, viz., if $K(H_1), K(H_2), \cdots, K(H_r)$ are identical in all respects [3.19], H is the rth power of H_i, written, $H \sim H_i^r$ $(i = 1, \cdots, r)$. Hence, $H_i, H_j, \cdots, H_k$ being any primitives, $H_i^sH_j^t\cdots H_k^l$ is defined when $s, t, \cdots, l$, are positive integers > 0; if $s = t = \cdots = l = 0$, the value is defined to be ϵ, any unit [3.10].

Similarly, $\psi\psi\cdots\psi$ (r factors) is defined to be ψ^r, for ψ as in 0.03. The enunciations of theorems and their proofs, concerning H_i forms, will be found in §§ 5, 6; and are deferred for reasons similar to those stated in 3.21. The majority of factorable functions in current use will be shown to be of the form $H_i^s\cdots H_k^l$, where $s, \cdots, l$ are positive or negative integers; the meaning of H_i^s for s negative is now defined similarly to the usual procedure in algebra:

*3.26. If H is an algebraic primitive form [3.23], a any positive non-zero integer, *any function H' which satisfies the equivalence $H^aH' \sim \epsilon$, where ϵ is any unit* [3.10], *is a value of H^{-a}, and H' is equivalent to ϵ/H^a, where ϵ/H^a is the ideal reciprocal of H^a*; written, $H' \backsim \epsilon/H^a$.

If ϵ is replaced in the foregoing by 1 (the absolute unit), then $1/H^a$ is the *pure reciprocal* of H^a; written $H' \sim 1/H^a$.

Also, similarly, from 3.25, for ψ as in 0.03, ϵ/ψ^a and $1/\psi^a$ are defined [cf. 3.07; note (1)]; also $\epsilon/\psi^a\psi^b\cdots$, is also obviously defined.

3.27. It is of course evident that the functional forms symbolized by ϵ/H^a, $1/H^a$, ϵ/ψ^a, $1/\psi^a$, do not *necessarily* exist.[1] In the following definitions,

[1] That is, without proof. It is doubtless possible to derive existence theorems from the definitions already given, but as this appears to be complicated, all such are postponed until after the introduction of several new concepts in §§ 4, 5, upon which all become practically obvious. Also, by deferring the proofs, sufficient material is provided for more than one dual of arithmetic [0.00 (i)].

the significance of the concepts defined is conditional upon the existence of ideal and pure reciprocals; viz., if these do not exist, the definitions are to be considered as meaningless. However, it is proved in § 5, that both kinds of reciprocals always exist, so that the theory which is being constructed is actual and not merely formal. These remarks apply also to 3.35, 3.36.

†3.28. Any product all of whose factors are units, is a unit. For, if ϵ_1, ϵ_2 are any units, $\Sigma_{(n)}\epsilon_1(d)\epsilon_2(n/d)$ [cf. 3.01 (i) for notation] consists of $\nu(n)$ [2.041 (v)] terms, each one of which vanishes if $n > 1$, for then either d or n/d is > 1. Also, if $n = 1$, the sum is $\Sigma_{(1)}\epsilon_1(1)\epsilon_2(1/1) \equiv 1$, [3.10]; hence $\epsilon_1\epsilon_2$ is a unit. From this and 3.08 it follows that ϵ_i $(i = 1, \cdots, r)$ being units, a_i $(i = 1, \cdots, r)$ a positive integer, $\epsilon_1^{a_1}\epsilon_2^{a_2} \cdots \epsilon_r^{a_r}$ is a unit.

†3.29. If ϵ is a unit, ψ any functional form [general, as in 0.06], $\epsilon\psi \sim \psi$. As in 3.28; in $\Sigma_{(n)}\epsilon(d)\psi(n/d)$, each term except that for which $d = 1$, vanishes; viz., the sum reduces to $\psi(n)$. Hence, etc.

*3.30. If in any equivalence, either of ψ, ψ' may replace the other (without affecting the validity of the equivalence), then ψ, ψ' are *identically equivalent:* symbolically $\psi \simeq \psi'$.

Identity and identical equivalence are distinct relations. A functional form ψ is identical only to ψ, and is identically equivalent to ψ, obviously; but if ψ is identically equivalent to ψ', it does not follow that ψ' is identical to ψ [cf. 3.31]. The distinction is of the greatest importance in connection with 0.00 (ii); by means of the concept of *associate functions* [3.32], uniformity is introduced into apparently heterogeneous masses of functions.

†3.31. $\epsilon\psi \simeq \psi$. [Notation and proof directly from 3.29; 3.30.]

*3.32. ϵ_i $(i = 1, \cdots, r, \cdots)$ being units, ψ any functional form, the identically equivalent functional forms $\epsilon_i\psi$ are *associates* of ψ.

†3.33. In respect to ideal multiplication a functional form is indistinguishable from any of its associates [3.30 to 3.32].

Hence, if $\psi \sim \psi_1\psi_2 \cdots \psi_r$ is a resolution of ψ into ideal factors, by 3.31, ψ_i may be replaced by $\epsilon_i\psi_i$ $(i = 1, \cdots, r)$, where ϵ_i is any unit, and by 3.02 (iii), 3.28, the result of such replacement is $\psi \sim \epsilon\psi_1\psi_2 \cdots \psi_r$ where ϵ is some unit, and the right of this equivalence is indistinguishable for purposes of ideal multiplication from $\psi_1\psi_2 \cdots \psi_r$.

3.34. *Henceforth, a functional form* ψ, *and its associates* [3.32] *when considered as ideal factors, will be considered as identical, and will be represented by* ψ. In this connection, cf. also 4.20; 5.21; 5.22. This procedure is in entire analogy with that in the usual Theory of Numbers; e. g., in connection with algebraic numbers. When it shall be necessary to distinguish a ψ from its associates, the same may be readily done.

*3.35. Units may be factors of a product, and yet be not explicitly in evidence, as, e. g., in the arithmetical definition of certain functions. If ψ, ψ', ψ'' are any functional forms, no one of which is a unit, such that $\psi \sim \psi'\psi''$, and if it be possible to resolve ψ', ψ'' further into (ideal) factors, $\psi' \sim \psi_1'\psi_2'$; $\psi'' \sim \psi_1''\psi_2''$, in such a manner that no one of ψ_1', ψ_2', ψ_1'', ψ_2''

is a unit, but some one of the products $\psi_1'\psi_1''$, $\psi_1'\psi_2''$, $\psi_2'\psi_1''$, $\psi_2'\psi_2''$ is equivalent to a unit, ϵ, then ϵ is a *latent unit in* $\psi'\psi''$.

Let $\epsilon \simeq \psi_1'\psi_1''$ be a latent unit in $\psi'\psi''$; hence (in accordance with 3.34) $\psi \sim \psi_2'\psi_2''$; and in $\psi_2'\psi_2''$ the latent unit ϵ is said to be *expressed* from $\psi'\psi''$.

Proceeding similarly with $\psi_2'\psi_2''$, let $\psi \sim \psi_3'\psi_3''$, in which a latent unit (if there be one) has been expressed from $\psi_2'\psi_2''$, giving $\psi_3'\psi_3''$. *If this process terminates*, giving, for r finite, $\psi \sim \psi_r'\psi_r''$, then $\psi_r'\psi_r''$ is the *expressed form* of $\psi'\psi''$, or of ψ. *The expressed form of $\psi'\psi''$ is that product identically equivalent to $\psi'\psi''$, in which there is no latent unit.*

It will be shown that the latent units in any product $\psi'\psi''$ may be expressed by one "twist of the press" [cf. 5.23; 5.24], and this applies to the following also.

From $\psi_1\psi_2\cdots\psi_r$ latent units are expressed by considering pairs of factors, as above; then in the result, pairs of factors, and so on, until if the process terminates, giving for s finite, $\psi_1\psi_2\cdots\psi_r \sim \psi_1^{(s)}\psi_2^{(s)}\cdots\psi_r^{(s)}$ wherein no pair of the factors $\psi_i^{(s)}$ $(i = 1, \cdots, r)$ contains a latent unit, $\psi_1^{(s)}\psi_2^{(s)}\cdots\psi_r^{(s)}$ is the *expressed form* of $\psi_1\psi_2\cdots\psi_r$.

*3.36. A functional form ψ whose expressed form [3.35] is divisible [3.07] by only ψ and units [3.10], is, if ψ is distinct from a unit, a *prime form;* and $\psi(n)$ is a *prime function.*

If H is a primitive form, which is also a prime form, H is a *prime primitive;* if H is in addition algebraic [3.23], then H is an *algebraic prime primitive*, etc.

3.37. With the exception of assigning a meaning to $\psi_1^{\psi_2}$ where ψ_1, ψ_2 are functional forms [this is done in 8.09 to 8.23], the definitions, etc., of § 3 provide sufficient material for the accomplishment of 0.00 (i) and (ii). But, in order to derive the properties of the ψ_i with a minimum of calculation, and also to make the processes and foundation sufficiently broad to support several duals of arithmetic, the further consideration of the ψ_i and their properties so far defined, is based upon the theory of sets, characteristics, and generators, the first two of which, themselves have an arithmetical theory,—and these will be defined and investigated in §§ 4; 5.

§ 4. Sets.

*4.00. Let t_i $(i = 1, \cdots, r)$ denote r positive, non-zero integers, no two of which are equal; and let $t_i > t_j$ when $i > j$. Then, the t_i when arranged in ascending order of magnitude, constitute an *integral set*, or, where there can be no confusion, a *set*, denoted by $| t_1, t_2, \cdots, t_r |$ or, by $| t |_r$. The *elements* of $| t |_r$ are the t_i; the *degree* is t_r, and the *extent*, r. Sets will occasionally be denoted by $S, S', \cdots, S_1, \cdots$, etc., where it is unnecessary to put the elements in evidence.

†4.01. If H is a primitive form, $I(H)$ is a set.[1] [3.17; 3.18.]

*4.02. Sets, S, S', are *identical*, $S = S'$, if each element of either is an element of the other. Non-identical sets, S, S', are *distinct;* $S \neq S'$.

[1] As elsewhere in the paper, the statement of a theorem is alone given, when the proof is obvious or a simple consequence of the definitions.

†4.03. The number of distinct sets of finite degree, t_r, is finite. The number of distinct sets of finite extent, r, is infinite.

*4.04. *Partition* of sets is defined as follows: Let $S = |\, t \,|_r$; $S' = |\, t' \,|_s$ be such that $t_1' > t_r$. Then [4.00], $t_j' > t_i$ $(i = 1, \cdots, r;\ j = 1, \cdots, s)$. Hence $|\, t_1, t_2, \cdots, t_r, t_1', t_2', \cdots, t_s' \,|$ is a set, S''. Then S, S' *together constitute a partition* of S''; written $S'' = S + S'$; or $S + S' = S''$. Inversely; if $1 \overline{\gtrless} s < r$, then $|\, t_1, t_2, \cdots, t_s \,|$ and $|\, t_{s+1}, t_{s+2}, \cdots, t_r \,|$ *together constitute a partition* of $|\, t_1, t_2, \cdots, t_r \,|$; written in either of the forms;

$$|\, t_1, t_2, \cdots, t_r \,| = |\, t_1, t_2, \cdots, t_s \,| + |\, t_{s+1}, t_{s+2}, \cdots, t_r \,|;$$

or,

$$|\, t_1, t_2, \cdots, t_s \,| + |\, t_{s+1}, t_{s+2}, \cdots, t_r \,| = |\, t_1, t_2, \cdots, t_r \,|.$$

*4.05. *Summation* of sets is defined as follows: Let $S' = |\, t' \,|_r$; $S'' = |\, t'' \,|_s$, and let t_1''', t_2''', $\cdots$, t_k''' be all the *distinct* integers chosen from among $t_1', t_2', \cdots, t_r'; t_1'', t_2'', \cdots, t_s''$, and let $t_1, t_2, \cdots, t_k$ denote the t_i''' $(i = 1, \cdots, k)$ arranged so that $t_i > t_j$ when $i > j$. Then, $|\, t \,|_k$ is the *summation* of $|\, t' \,|_r$ and $|\, t'' \,|_s$; written, $|\, t \,|_k = |\, t' \,|_r + |\, t'' \,|_s$; or, $|\, t' \,|_r + |\, t'' \,|_s = |\, t \,|_k$.

*4.06. The laws of partition and summation of sets together constitute the laws of *addition* of sets.

4.07. It is scarcely necessary to remark that *as a relation,* $=$ has distinct meanings in 4.02; 4.04; 4.05. In 4.02, $=$ is the sign of a certain relation between sets; in 4.04, $=$ and $+$ *together* are the sign of a *single* relation between three sets of a particular kind; in 4.05 $=$ and $+$ are together the sign of a single relation between sets of a particular kind, and, in general, it is obvious that ($=$ and $+$) has distinct meanings, although similar, in 4.05; 4.04. Like remarks apply to some of the sections following; in no case is there cause for confusion, and this repeated use of certain signs obviates the introduction of new symbols.

†4.08. Addition of sets is commutative and associative.

*4.09. With the notations of 4.04; 4.05, respectively; S is the *difference* of S'' and S'; S' is the *difference* of S'' and S; each of $|\, t' \,|_r$, $|\, t'' \,|_s$ is the *difference* of $|\, t \,|_k$ and the other. The difference of S_1 and S_2 is written $S_1 - S_2$. Note that a pure negative, $- S$, is not defined; but cf. 4.53.

*4.10. Clearly, $S - S$ is thus far without significance, it is denoted by $|\, 0 \,|$, and is defined to be the *identity set.* With respect to addition of sets, $|\, 0 \,|$ is to have the formal properties of 0 in relation to addition of integers; e. g., $S \pm |\, 0 \,| = S$; $|\, 0 \,| \pm |\, 0 \,| = |\, 0 \,|$, etc. [cf. 4.21], combining:

†4.11. With respect to addition (of sets) all sets form a commutative group. The identity element in the group is $|\, 0 \,|$.

*4.12. Let ϵ, ϵ', $\cdots$, ϵ_1, $\cdots$, denote quantities each one of which can assume only the value 0 or the value 1; and let t_i', t_j'' denote positive, non-zero integers; then $\epsilon' t_i' + \epsilon'' t_j''$ represents three positive, non-zero integers, viz., t_i', t_j'', $t_i' + t_j''$. Considering $|\, t' \,|_r$, $|\, t'' \,|_s$, let t_1''', t_2''', $\cdots$, t_k''' denote all the distinct positive non-zero integers represented by $\epsilon' t_i' + \epsilon'' t_j''$ $(i = 1, \cdots, r;$

$j = 1, \cdots, s$), and let $t_1, t_2, \cdots, t_k$ denote the t_i''' ($i = 1, \cdots, k$) arranged in ascending order of magnitude; then, the *product* of $| t' |_r$, $| t'' |_s$ is $| t |_k$; and $| t' |_r$, $| t'' |_s$ constitute a *pair of factors* of $| t |_k$; in symbols, $| t' |_r \, | t'' |_s = | t |_k$; or, $| t |_k = | t' |_r \, | t'' |_s$; and $| t |_k$ is *factorable* by each of $| t' |_r$, $| t'' |_s$. This defines *multiplication* for sets. E. g.,

$$| 1, 2, 3, 4, 5 | \, | 1, 2, 3, 4, 5, 6 | = | 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 |.$$
$$| 2, 5 | \, | 2, 4, 6 | = | 2, 4, 5, 6, 7, 8, 9, 11 |.$$

†4.13. Multiplication for sets is commutative.

†4.14. Multiplication for sets is associative.

For, $(| t' |_a \, | t'' |_b) | t''' |_c$, $| t' |_a (| t'' |_b \, | t''' |_c)$ each has as elements all the distinct non-zero values of $\epsilon' t_i' + \epsilon'' t_j'' + \epsilon''' t_k'''$ ($i = 1, \cdots, a$; $j = 1, \cdots, b$; $k = 1, \cdots, c$).

†4.15. Multiplication for sets is distributive with respect to addition of sets; viz., $| t' |_a (| t'' |_b + | t''' |_c) = | t' |_a \, | t'' |_b + | t' |_a \, | t''' |_c$. This follows similarly to 4.14, on referring to 4.06.

†4.16. The degree of the product of two sets is the sum of the degrees of the sets. For, $| t' |_r \, | t'' |_s$ contains as its greatest element, $t_r' + t_s''$; and t_r', t_s'' are respectively the greatest elements of $| t' |_r$, $| t'' |_s$.

†4.17. The extent of the product of two sets is greater than the extent of either of the two sets. For, $| t' |_r \, | t'' |_s$ contains $t_r' + t_s''$ which is in neither of $| t' |_r$, $| t'' |_s$, also, it contains all members of either.

†4.18. No sets S, S' (as defined in 4.00) exist such that $S = SS'$ [4.16].

*4.19. Assuming that $| a_1, a_2, \cdots, a_r |$ obeys the laws of multiplication for sets, and is such that, S being any set, $S \, | a_1, a_2, \cdots, a_r | = S$, the symbol $| a_1, a_2, \cdots, a_r |$ is defined to be a *unit set of extent r.*

†4.20. Every element a_i ($i = 1, \cdots, r$) in a unit set of extent r, is 0 [4.16]. Hence [cf. 4.10], $| 0 | \, S = S$. With respect to multiplication for sets, unit sets of different extents, are indistinguishable, and may be each replaced by $| 0 |$.

4.21. In 4.10, 4.19, 4.20, $| 0 |$, $| 0, 0, \cdots, 0 |$ are *not* [by 4.00] *sets;* but there is no contradiction in referring to them as *unit sets*, etc., regarding this as a new term, etc. Unit sets of extent > 1, are not strictly necessary to 0.00 (ii); they have been defined for reasons given in 4.64. Henceforth, with respect to multiplication of sets, $| 0 |$ alone is taken as the identity element. Thus, the present theory of sets has but one unit [cf. 0.01 (i); 4.64]. It is important to note that:

†4.22. With respect to multiplication for sets, sets do not form a group. For, in general, multiplication has not a unique inverse; that is, if S, S' are given sets, a set S'' may be determined in several ways so that $S = S'S''$. It will be sufficient to show that S'' may be found in two ways to satisfy $| 2, 3, 4, 5, 6, 7, 8, 9, 10 | = | 3, 4, 5 | \, S''$. Here, $S'' = | 2, 4, 5 |$, or $| 2, 3, 4, 5 |$; as may be verified directly by multiplication.

*4.23. If a set S admits as factors only the single pair $| 0 |$, S, then S is a *prime set*, or simply a *prime*. A set that is not prime is *composite*.

†4.24. Primes exist. For, $| t |_1$ is prime [by reductio ad absurdum from 4.16 or 4.17].

†4.25. The number of primes is infinite [4.24; cf. also 4.62].

†4.26. A composite set of either finite degree or finite extent, provided that in the latter case the degree is not infinite, is the product of distinct pairs of sets in only a finite number of ways [from 4.16; 4.17].

†4.27. The product of two primes may be factorable by a prime distinct from either of the two. It suffices to find a single example in which this is true. Clearly, each of $| 1, 2 |$, $| 4, 5 |$ is prime; their product is $| 1, 2, 4, 5, 6, 7 |$, which is also the product of $| 1 |$ and $| 1, 4, 5, 6 |$; and $| 1 |$ is prime.

4.28. It is now required to devise a definition of divisibility for sets which shall restore to the theory of sets the fundamental theorem of arithmetic, stated in 0.00 (i), and which shall result in a *unique* quotient in every case, $S_1 \div S_2$. The method adopted admits of wide extension and application, it is clearly suggested by many well-known and important processes in arithmetic, particularly in the Theory of Algebraic Numbers. First, an analytical definition is given, and then the processes of addition, subtraction, multiplication, and division for sets are exhibited on a lattice of unit squares, as the actual processes are best performed graphically. The lack of a short method for resolving a set into its prime divisors, is but another point of resemblance between the theory of sets and arithmetic. It will be seen when the geometrical representation is examined, that certain of the following definitions are more inclusive than is necessary for the investigation of primitives [3.14], to which sets are presently applied [§ 5]; this is accounted for in 4.64, and in any case there is less difficulty in devising a general concept of divisibility than there is in evolving a definition to fit only a particular class of elements. As always, the unit set, $| 0 |$, is written in explicit form; viz., S, $| t |_r$, $\cdots$, etc., never denote $| 0 |$.

*4.29. Let $S = S_i S_j'$ $(i, j = 1, \cdots)$ denote all the resolutions of a composite S of finite degree into a pair of factors, S_i, S_j'. By 4.16; 4.03 the number of such *distinct* pairs of factors is finite. Among all pairs S_i, S_j'; S_j', S_i; $\cdots$ will be some in which the degree of S_i is less than or equal to the degree of either factor in any other pair, and all these S_i of lowest degree, constitute the *minimal class of divisors with respect to* S, denoted by $[S_0]$.

†4.30. With the notation of 4.29; each member of $[S_0]$ is prime. For, if not, let $S_0^{(i)} = S^{(k)}S^{(l)}$ be a composite member of $[S_0]$. Then $S = S^{(k)}(S^{(l)}S_j')$; whence $S^{(k)}$, of lower degree than $S_0^{(i)}$ [4.16], is a factor of S; hence $[S_0]$ is not the minimal class of divisors with respect to S since $S^{(k)}$ is not a member of $[S_0]$; a contradiction.

†4.31. The class $[S_0]$ contains only a finite number of members. By 4.03, since all members of $[S_0]$, by 4.29 are of the same degree.

*4.32. Each member of $[S_0]$ is a factor of S [4.29]. This will be expressed by: *S is divisible by* $[S_0]$; symbolically, $S = [S_0]S''$; which may be regarded as equivalent to $S = S_0^{(i)}S_j''$ $(i, j = 1, \cdots)$ wherein $S_0^{(i)}$ $(i = 1, \cdots)$ represents successively each member of $[S_0]$. With this notation:

†4.33. If $S = [S_0]S''$, all the sets S_j'', in number finite, are of the same degree [4.29; 4.31; 4.16]. Also,

*4.34. With respect to a particular S_j'' there is a minimal class of divisors; let all such minimal classes be denoted by $[S_j'']$ $(j = 1, \cdots)$. Defining the degree of $[S_j'']$ as the degree of any one of its members, among all the $[S_j'']$ $(j = 1, \cdots)$ there will be certain whose degree is equal to or less than the degree of any $[S_j'']$. All the members of all these minimal classes of lowest degree may be put in a single class, $[S_1]$; the *minimal class of divisors of* S *with respect to* $[S_0]$; and, obviously,

†4.35. Every $[S_0]S_1^{(i)}$, where $S_1^{(i)}$ $(i = 1, \cdots)$ are the members of $[S_1]$, is a factor of S. This will be expressed by;

*4.36. S *is divisible by* $[S_0][S_1]$; or, $S = [S_0][S_1]\bar{S}$, where $\bar{S}$ may be $= |\, 0\, |$.

4.37. If in 4.36, $\bar{S}$ is different from $|\, 0\, |$, the processes and definitions of 4.32 to 4.36 may be continued in an obvious way, until finally an $\bar{S}$ identical with $|\, 0\, |$ is reached, and $S = [S_0][S_1]\cdots[S_g]$; and

†4.38. g is finite [4.16]. Each member of $[S_i]$; $0 \leqq i \leqq g$ is a prime, and if $D[S_i]$ is the degree [4.34] of $[S_i]$,

$$D[S_1] \geqq D[S_0];\ D[S_2] \geqq D[S_1];\ \cdots,\ D[S_g] \geqq D[S_{g-1}].$$

Also, each $[S_i]$ contains only a finite number of sets.

*4.39. If each member of $[S_i]$ is a member of $[S_j]$, and if each member of $[S_j]$ is a member of $[S_i]$, $[S_i]$ and $[S_j]$ are *identical;* $[S_i] = [S_j]$. If $[S_i] = [S_1] = [S_2] = \cdots = [S_r]$, then $[S_1][S_2]\cdots[S_r]$ is represented by $[S_i]^r$, and in this notation there is *the resolution into prime divisors* $S = [S_i]^{r_i}[S_j]^{r_j}\cdots[S_k]^{r_k}$, of any composite set S of finite degree, each of the $[S_i]$, $[S_j]$, $\cdots$, $[S_k]$ being a *prime divisor.* The degree of S is denoted by $D(S)$; and clearly;

†4.40. $D(S) = r_iD[S_i] + r_jD[S_j] + \cdots + r_kD[S_k]$.

*4.41. If in the resolution in 4.39, from $[S_i]$, $[S_j]$, $\cdots$, $[S_k]$ there be selected the respective classes of sets whose extent in each case is a minimum, the resulting classes are called *principal,* and are denoted by $\{S_i\}$, $\cdots$ etc., viz., $\{S_i\}$ contains all members of $[S_i]$ whose extent is equal to or less than the extent of any member of $[S_i]$, and $S = \{S_i\}^{r_i}\{S_j\}^{r_j}\cdots\{S_k\}^{r_k}$ is the *principal resolution of* S *into prime divisors.* Note that *these are not logical products of the classes;* such are considered in another connection [§ 9].

4.42. In an obvious sense that need not be dwelt upon, especially in view of its analogies to certain parts of the Theory of Numbers that will at once suggest themselves, the resolutions given respectively in 4.39; 4.41, are *unique factorizations of a given set.* By *the* resolution of S, the *principal* resolution will henceforth be meant.

*4.43. The *divisors of* S, are, in the notation of 4.41;

$$\{S_i\}^{r_i'}\{S_j\}^{r_j'}\cdots\{S_k\}^{r_k'} \text{ where } 0 \leqq r_i' \leqq r_i;\ 0 \leqq r_j' \leqq r_j;\ \cdots;\ 0 \leqq r_k' \leqq r_k;$$

$|\, 0\, |$ being the divisor when $r_i' = 0$; $r_j' = 0$; $\cdots$; $r_k' = 0$.

*4.44. S_1 *is divisible by* S_2 if every divisor of S_2 is a divisor of S_1.

*4.45. By means of a few obvious changes in the statements, from 1.01 is derived a definition of *simple sets;* $|\,0\,|$ is considered as simple. Similarly, the whole of § 1 may be rewritten upon sets, instead of upon integers, as a basis. In particular,

†4.46. Any set may be resolved into distinct simple sets in one way only. More generally;

†4.47. To any theorem regarding simple numbers there is a unique correspondent in the theory of sets, obtained on replacing *number* throughout by *set.*

The interpretation of the results is another matter; it is related to the problems of analysis situs mentioned below, but, not being essential to 0.00 (i) or (ii), is not further considered here. But (cf. also 4.61–3.),

†4.48. Four distinct arithmetical theories exist for sets. The four are distinguishable from one another according to the respective definitions adopted for divisibility, which may be as in 4.43, or directly from 4.39 in a manner similar to 4.43, and in either case there are the alternatives of prime or simple sets as a basis. For a more general theorem, cf. 6.37.

4.49. The four theories of 4.48 are all similar; more precisely, they are simply isomorphic.[1] Although so like in appearance, the respective results to which they give rise when applied to the primitives, are totally distinct in kind. One only is carried further in this discussion; that in which divisibility is as defined in 4.43. A very brief account of the lattice representation of the chief concepts in 4.00 to 4.47 is now given. The necessary definitions have been so cast that the proofs of the validity of the several processes in relation to sets, are self-evident; details, and the enunciations of the many theorems suggested by symmetry, etc., in the diagrams, may be left to the reader [cf. Figs. 1, 2].

*4.50. Let m, n each assume in turn all positive integral values from 0 to ∞; the points (m, n) lie in the *positive quarter plane* XOY; OX, OY being the axes of coordinates, chosen rectangular for convenience; the (m, n) are the *lattice points.* A set, $|\,t\,|_r$, is represented in either of two ways upon XOY:

(i) The *X-representation;* consisting of the half-lines $\{x = t_i\}$ $(i = 1, \cdots, r)$, a *half-line* being that segment of any straight line which lies wholly *within* XOY. By *within* is always meant *the interior of a region and the boundaries of that region.* In the diagram, the half lines $\{x = t_i\}$ are to be dotted, or *light.* The lattice points lying on $y = 0$, and which do not lie to the right of $(t_r, 0)$, are $(m, 0)$, for $m = 0, 1, \cdots, t_r$. Through each of the $(m, 0)$ which does not lie upon a half-line, $\{x = t_i\}$, is drawn a full, or *dark* half-line, parallel to $x = 0$. These are the *dark lines* of $|\,t\,|_r$.

(ii) The *Y-representation,* is obtained by rotating the X-representation through an angle of $-\pi/2$, and then through π about OX. The X set is read

[1] The precise distinction between theories as simply or multiply isomorphic, based upon the correspondences between the fundamental relations connecting their respective elements, is made in the second part. All theories of this first part are only simply isomorphic to each other and to the arithmetic of integers; although, from 6.33 to 6.37 may be derived theories multiply isomorphic to arithmetic.

in the direction OX. In either representation, the totality of light and dark lines clearly depicts $|t|_r$ unambiguously, and the following is evident:

†4.51. *Addition.* The X representations of $|t'|_r$, $|t''|_s$ are made simultaneously upon XOY. The coincidence of two half-lines is to be represented by a light line except in the case when the two lines are both dark, when the coincidence is to be dark; all lines of either set that coincide with no line of the other are to be left as they are. The total result is $|t'|_r + |t''|_s$.

†4.52. *Subtraction.* To represent $|t'|_r - |t''|_s$, *when this exists* [cf. 4.09]. Proceed as in 4.51, except that *all* coincidences are to be dark.

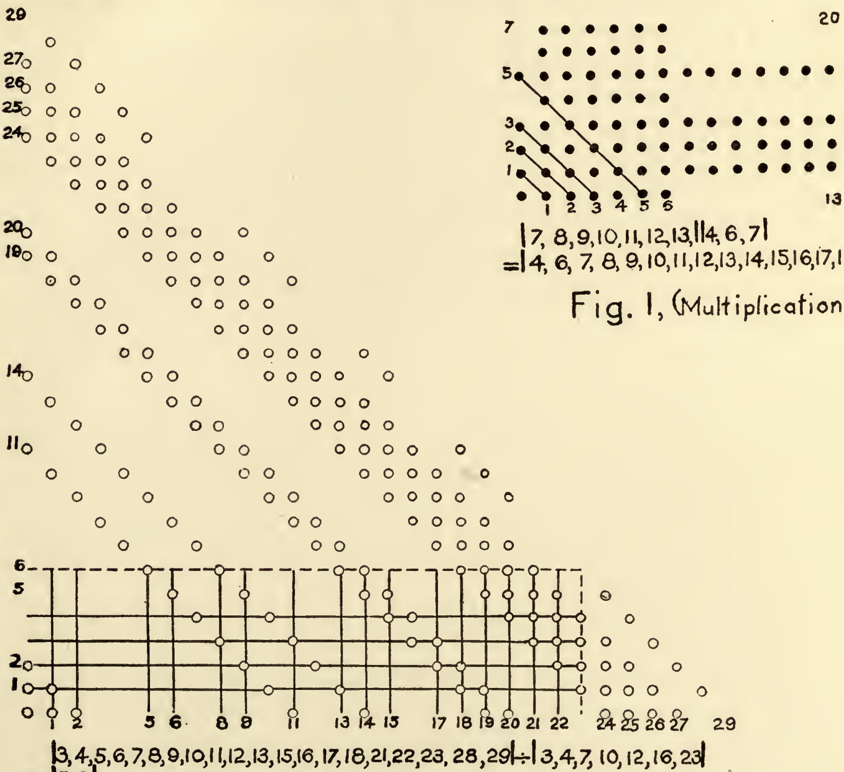

Fig. 1, (Multiplication).

Fig. 2, (Division).

4.53. A *pure negative*, $-S$, may be defined by extending 4.50 to include all of the upper half plane, and $-S$ is the reflection in $x = 0$, of S. Also, $S - S = |0|$ is the half-line $\{x = 0\}$, etc., and the properties of pure negatives may be developed as in algebra. Pure negatives have not been considered because, in relation to primitives, they have no significance [but, cf. § 12].

*4.54. *(i) A lattice point which lies upon a dark line is a *node* (in fig. 1, heavy dots). Nodes will sometimes be indicated by small circles; and a line which passes through nodes only by the nodes on it; this in order to keep the diagram simple. By convention, (0, 0) is always a node.

*(ii) The half-lines $\{x + y = k\}$, for $k = 0, 1, 2, \cdots, \infty$, are *diagonals*. The diagonal $\{x + y = 0\}$ is clearly (0, 0). The *kth diagonal* is $\{x + y = k\}$.

*(iii) That segment of any line which lies within [cf. 4.50 (i)] a closed region which is everywhere convex, is a *sect;* a *diagonal sect* is obviously defined for a given closed region everywhere convex.

*(iv) A sect is *proper* or *improper* according as all the lattice points upon it are not, or are, nodes.

*(v) The half-line $\{nx - my = 0\}$ (m, n, + integers), intersects every diagonal sect in the rectangle whose vertices are $(0, 0)$, $(m, 0)$, (m, n), $(0, n)$. Starting from (0, 0) and proceeding along $\{nx - my = 0\}$ to (m, n), the diagonal sects are numbered successively, $0, 1, 2, \cdots, (m + n)$; and j, where $0 \leqq j \leqq (m + n)$, is the *suffix* of the jth diagonal sect, and is *proper or improper* according as the diagonal sect is proper or improper.

†4.55. *Multiplication.* The X-representation of $|\,t'\,|_r$, and the Y-representation of $|\,t''\,|_s$ are made simultaneously upon XOY (or, vice versa); and $t_r' \equiv m;\ t_s'' \equiv n$. [Cf. 4.54 (v).] The product, $|\,t'\,|_r|\,t''\,|_s$ is obtained by writing down successively the *proper* suffixes of the diagonal sects in order, proceeding as in 4.54 (v). The proof is immediate from the definitions in 4.12; 4.54; noticing that on the jth diagonal sect the sum of the coordinates of any lattice point is j. In Fig. 1, the nodes only are shown on the representations.

4.56. *Division.* There are two cases: (i) where it is known that the set S_2 is factorable by the set S_1, it is required to to find *all* sets S such that $S_2 = SS_1$. (ii) Where it is required to determine whether an S exists such that, for S_1, S_2 given, $S_2 = SS_1$; in other words, to resolve S into its prime divisors. These are in order of difficulty, but both are simple for actual examples if the degree is low.

†4.57. *Division; Case* (*i*) [4.56]. (i) To find a value of $|\,t\,|_g$ such that $|\,t'\,|_r = |\,t''\,|_s|\,t\,|_g$, and (ii) To find all such values.

(i) Both the X- and Y-representations of $|\,t'\,|_r$ are made simultaneously upon XOY, and then everything except the diagonal sect (proper) $\{x + y = t_r'\}$ and the improper diagonal sects obtained by surrounding every lattice point on the join of $(n, 0)$, $(0, n)$, $0 \leqq n < t_r'$ and $(n, 0)$, $(0, n)$ *nodes* in the respective representations, by a small circle, is erased. This may obviously be done in one step, without erasure; see Fig. 2.

The diagram now consists of a right triangle $(0, 0)$, $(t_r', 0)$, $(0, t_r')$, crossed by diagonal sects (represented by successions of small circles) parallel to $\{x + y = t_r'\}$. Complete the rectangle of which (0, 0) and $(t_s'', t_r' - t_s'')$ are opposite vertices. Through each node of the X-representation of $|\,t''\,|_s$ which (node) lies upon $\{y = 0\}$ draw a dark sect within the rectangle, parallel to $\{x = 0\}$. Certain nodes of the improper diagonal sects will lie upon the sect $\{x = t_s''\}$. Through each of these draw a dark sect, parallel to $\{y = 0\}$ (within the rectangle). Examine the rectangle to see if any node on the diagonal sects of the right triangle does not lie upon one of the dark sects that have been drawn parallel to the axes. Through *every such* node draw a dark

sect *parallel to* $\{y = 0\}$. Then, a value of $| t |_g$ is found by reading off in order from (0, 0) to $(0, t_r' - t_s'')$ on $\{x = 0\}$, the lattice points which are not nodes.

The proof will be evident on comparing the rectangle with that in 4.55; also,

(ii) All possible values of $| t |_g$ are obtained by drawing dark sects as in (i) [after "Examine," etc.] in such a way that *no new diagonal sect, all of whose lattice points are nodes, shall be added to those already in the right triangle.* For, if any such new diagonal sect be added, the set $| t' |_r$ is changed by the omission of an element, and if no such new diagonal sect be added, $| t' |_r$ is still represented (on the right triangle). Cf. 4.55. Any way of drawing the dark sects as just described, gives a value of $| t |_g$, read off as in (i); the distinct values of $| t |_g$ so obtained give the complete solution of the problem.

Obviously, if the degrees of the sets are large, the *geometrical* problem of finding all the $| t |_g$ becomes very difficult; cf. 4.59. It may be remarked that these processes will only be thoroughly understood by the working out in detail of a few examples, which should be constructed first by 4.55 ; the like applies to:

†4.58. *Division; Case* (*ii*) [4.56]. To resolve $| t |_r$ into its prime (set) divisors. The right triangle is drawn as in 4.57, with its improper diagonal sects, etc., and in addition, the node formed by the intersection of each dark line with the sect $\{x + y = t_r\}$ is indicated by a small circle round the corresponding lattice point. These may be called the *vertical nodes*, for a reason that will appear presently. On $\{x + y = t_r\}$, choose any lattice point as a vertex, and complete the rectangle whose opposite vertex is (0, 0). Clearly, if the chosen lattice point is a vertical node, the rectangle, for no system whatever of dark lines drawn as in 4.57 (ii), can represent (as in 4.55) a pair of factors of $| t |_r$; for, in this rectangle, the diagonal sect $\{x + y = t_r\}$, viz., the vertical node, is improper. Choose, therefore, that lattice point on $\{x + y = t_r\}$ which lies *nearest* the X-axis, and which is *not* a vertical node, and proceed as just above. If there be more than one quotient, obtained as in 4.57 (ii), proceed similarly with each; and so on. In this way 4.29 to 4.39 is performed graphically; and obviously, the principal resolution [4.41] may be read off in each case by inspection.

4.59. The whole of division, and resolution into prime factors, may be performed in either half of the right triangle used, which is bisected by $\{y = x\}$. This requires less paper, but more patience. Obviously, the essential *geometrical* character of a set is preserved if the lattice and its lines, diagonals, etc., be distorted in any way, provided, however, that the points of intersection of the various lines with any one of them are preserved in the same order. The question of being able to draw in the diagonals as required in 4.57, 4.58, and of the number of ways in which this may be done, may clearly be formulated in more than one way as a problem of analysis situs (or, possibly, more properly, topology), of the same genus as some considered by

Euler,[1] and whose complete solution depends upon the analytical theory of sets. This may be developed by anyone who is interested; it is intimately connected with the theory of *sums of greatest integers*,[2] viz., sums of expressions $[m/n]$, etc., and with the arithmetical integration,[2] mentioned in 8.25.

4.60. A theorem from the elements of the Theory of Assemblages is used in the sequel so frequently that it may be stated for reference: Let $\{A, A', A'', \cdots\}$ denote a denumerable assemblage whose elements A, A', A'', $\cdots$ are all either finite or denumerable assemblages; then, the assemblage of the logical sum of the elements of $A, A', A'', \cdots$, is denumerable.

†4.61. The assemblage of sets is denumerable. [From the definitions relating to sets and 4.60.]

†4.62. The assemblage of primes (viz., prime *sets*) is denumerable. For, first, this assemblage contains $| t |_1$ $(t = 1, 2, \cdots, \infty)$; second, it is a proper part of the assemblage of sets.

†4.63. The assemblage of minimal classes of divisors is denumerable.

4.64. It may now be seen in what sense the sets constitute an arithmetical field, and hence, in what respects there is an arithmetical theory for sets. The theory is obviously an extended arithmetical theory in the same sense that the theory of algebraic numbers is; for, the minimal classes of divisors are not sets, that is, the *prime* elements are not in the original data of the theory, regarding those data as sets, but the primes are *classes* of sets. This corresponds in an obvious manner to the introduction of ideals in connection with the theory of algebraic numbers; for, it was shown [4.27] that an essential property of arithmetical division is not valid for factors, but is restored on the broader basis of classes, which in a definite way includes the special case of factorization. Or, by a slight change ab initio in the point of view, an arithmetical field, and hence, theory, in the strict sense of 0.01 may be easily imagined upon the sets as a basis, by considering each set as the limiting case of a class of sets; viz., the limiting class is to contain but a single member. This is in analogy to the principal ideals of arithmetic, in which, only by a similar change in the usual meanings of the words (viz., in the arithmetic of integers) can an *integer* be called a unique product of prime ideal factors. The theory of sets is not considered further, except in its relations to the primitives. It is now apparent that 4.00 to 4.63 is but the simplest case of a more general theory of sets, in which the *elements* of a set are themselves sets, which in turn is included in the case in which the elements of these set-elements are again sets, and so on. In the general case, the lattice is in n-space. There seems to be no example at present in arithmetic of these more general numerical functions corresponding to the n-space lattice.

4.65. $S \equiv S' \bmod S''$, is; "$S - S'$ exists and is divisible by S''.*"

[1] Euler: *Solutio problematis ad Geometriam situs pertinentis.* Berol. Acad. Sci., 1759. The writer has not had an opportunity of consulting this paper during the last two years, and the reference is taken from Lucas: *Théorie des Nombres*, p. 96.

[2] The reader who wishes to pursue the subject may consult J. Hacks: Acta Mathematica, t. xvii (1893).

4.66. There is a theory of congruences for sets; also a theory of forms; but as these belong properly to the second part, when the congruence of functions is considered, they are deferred until then. But it is clear that to any S modulus there is a finite complete system of residues, etc.

§ 5. Characteristics; Generators.

*5.00. The base of a primitive has been defined, 3.18. More generally, if it be possible to establish a (1, 1) correspondence between the positive, non-zero integers and the members of an assemblage of functional forms, f_1, f_2, $\cdots$, f_r, $\cdots$, in such a way that r and f_r are correspondents, the concept of a base may be extended, thus; let t_i and f_i' be correspondents, where t_i is an element of $|\, t\,|_r$, and f_i' is an f_j $(j = 1, 2, \cdots, r, \cdots)$; then, the f_i', when arranged in the order f_1', f_2', $\cdots$, f_r', constitute the functional set, $[f']_r$; and $|\, t\,|_r$ and $[\,f']_r$ are *corresponding sets*, or simply, *correspondents*.

5.01. Throughout § 5, H's denote primitive forms, [3.14; 3.17]; also, cf. 4.01, 3.18, $B(H)$ is $[h]_r$, $I(H)$ is $|\, t\,|_r$, [4.00], whence, for $i = 1, \cdots, r$, the elements h_i, t_i correspond. K is defined in 3.19. For 0.00 (i) and (ii), the laws of combination of symbols $[\,f\,]_r$, [5.00], presently given, are fundamental. The introduction of generators is merely a convenience; all proofs may be given without their aid, but at much greater length; here, the theorems become self-evident. Also, cf. 3.22.

†5.02. Each of the sum, product, or difference, the last being as defined in 4.09 but not as in 4.53, of two integral sets, being again an integral set is, obviously the index of at least one primitive; directly from 3.14, 3.18, 4.01. Clearly also, the functional sets corresponding [5.00] to each of these combinations of integral sets, as yet are arbitrary; they will therefore be defined in a manner appropriate to *ideal* multiplication and addition, the last being wholly distinct from algebraic addition [cf. 5.33].

*5.03. Let $[\,f']_r$, $|\, t'\,|_r$ be correspondents; likewise $[\,f'']_s$, $|\, t''\,|_s$ and $[\,f\,]_g$, $|\, t\,|_g$; also let the integral sets be such that $|\, t'\,|_r + |\, t''\,|_s = |\, t\,|_g$; then $[\,f\,]_g$ is defined[1] to be $[\,f']_r + [\,f'']_s$ where, for $i = 1, \cdots, g$,

(i) $f_i = f_i'$ if t_i is in $|\, t'\,|_r$ but not in $|\, t''\,|_s$;

(ii) $f_i = f_i''$ if t_i is in $|\, t''\,|_s$ but not in $|\, t'\,|_r$;

(iii) $f_i = f_i' + f_i''$ if t_i is in $|\, t'\,|_r$ and in $|\, t''\,|_s$.

The possibilities are evidently exhausted, as, by 4.04, 4.05, t_i must be either in $|\, t'\,|_r$ or in $|\, t''\,|_s$, or in both, and cannot be in neither. The result of adding the functional sets may be written [cf. 4.07], $[\,f\,]_g = [\,f']_r + [\,f'']_s$, or similarly, with the members transposed, which implies, as for sets, that the latter also is part of the definition; the latter is to apply to subsequent definitions and need not be stated explicitly.

†5.04. Addition for functional sets is commutative and associative.

*5.05. If in 5.03 the sign of every f_i'' be changed, the result defines $[\,f']_r - [\,f'']_s$. This exists only if $|\, t'\,|_r - |\, t''\,|_s$ is as in 4.09.

[1] $f_i' + f_i''$ is to be considered as abstracted from $f_i'(n) + f_i''(n)$, by an obvious extension of "abstraction" as defined in 0.06.

*5.06. With the notation of 5.03, let *all* the solutions of $t_i' + t_j'' = t_l$ $(1 \leqq l \leqq g)$ be given, for t_i', t_j'' respectively members of $| t' |_r$, $| t'' |_s$, and l fixed, by $t_a' + t_x'' = t_l$; $t_b' + t_y'' = t_l$; $\cdots$; $t_c' + t_z'' = t_l$; and write [cf. 3.12]:

$$| f_l''' | \equiv | f_a' f_x'' | + | f_b' f_y'' | + \cdots + | f_c' f_z'' |;$$

and let a (1, 1) correspondence be established between the t_l, f_l as follows:

(i) $f_l = | f_l''' | + f_l'$ if t_l is in $| t' |_r$ but not in $| t'' |_s$;

(ii) $f_l = | f_l''' | + f_l''$ if t_l is in $| t'' |_s$ but not in $| t' |_r$;

(iii) $f_l = | f_l''' | + f_l' + f_l''$ if t_l is in $| t' |_r$ and in $| t'' |_s$;

then, as in 5.03 the possibilities are exhausted, and by 5.00 the f_l form a functional set $[f]_g$, defined to be $[f']_r[f'']_s$, the *product* of $[f']_r$, $[f'']_s$; or, $[f']_r[f'']_s = [f]_g$. This defines *multiplication* for functional sets; and,

†5.07. Multiplication for functional sets is commutative, associative and, with respect to addition as defined in 5.03, distributive.

5.08. The proofs of 5.04, 5.07 may be seen immediately from the definitions, or intuitively from 5.13. Multiplication for sets is basic for ideal multiplication [§ 3], and prepares the way for establishing an isomorphism between arithmetic (in toto), the arithmetical theories of this paper, and the theory of polynomials in two variables, each with all; and more generally, on the obvious extension of 5.06 is set up (in the second part), such an isomorphism between the theory of rational algebraic functions and those mentioned in 4.64.

*5.09. Conversely to 5.06, if $[f]_g$, $[f']_r$ are given, and $[f'']_s$ exists such that (i): $[f']_r[f'']_s = [f]_g$, then $[f'']_s$ is *a value of the quotient of* $[f]_g$ *by* $[f']_r$. In general, $[f'']_s$ is not unique, and it may be taken as the symbol of the entire class of functional sets satisfying (i); and any relation involving $[f'']_s$ is considered to be significant only if the relation is valid for *every member of the class denoted by* $[f'']_s$.

*5.10. Referring to 5.01, 5.03, 5.05, 5.06, the right-hand members of (i) to (iii) define the left:

$$\text{(i)}\quad K(H_1) + K(H_2) = \begin{bmatrix} I(H_1) + I(H_2) \\ B(H_1) + B(H_2) \end{bmatrix};$$

$$\text{(ii)}\quad K(H_1) - K(H_2) = \begin{bmatrix} I(H_1) - I(H_2) \\ B(H_1) - B(H_2) \end{bmatrix};$$

$$\text{(iii)}\quad K(H_1)K(H_2) = \begin{bmatrix} I(H_1)\,I(H_2) \\ B(H_1)B(H_2) \end{bmatrix}.$$

(iv) If $K(H)$ exists such that $K(H)K(H_1) = K(H_2)$, then $K(H)$ is a value of $K(H_2)/K(H_1)$, etc. (as in 5.09). There is in (i) to (iii) the tacit assumption that the right hand members *are* characteristics; this is true obviously from the several definitions, the corresponding primitive being in any case uniquely defined. If in (iv) *no* $K(H_1)$ exists having the required property, viz., that $K(H_2)/K(H_1)$ be a characteristic, then $K(H_2)$ is *prime.* The further consideration of primitives, characteristics, etc., is much simplified by the introduction of

*5.11. *Generators.* Let $I(H) = |\,t\,|_r$; $B(H) = [h]_r$, and denote by z, x independent variables; *with H is associated the generator*, $\Gamma(H)$, where

$$\Gamma(H) \equiv h_1(x)z^{t_1} + h_2(x)z^{t_2} + \cdots + h_r(x)z^{t_r};$$

and $1 + \Gamma(H)$ is denoted by $\gamma(H)$. For *all* primitives, the independent variables in the associated generator are to be x, z; $\gamma(H)$ *is the generating function of* H.

†5.12. The difference of identical generators being excluded, it follows from 5.11 and the definition of a primitive, that to each of the sum, difference or product of two generators is associated a unique primitive; also,

†5.13. (i) If $K(H_1) + K(H_2) = K(H)$, then $\Gamma(H_1) + \Gamma(H_2) = \Gamma(H)$.
(ii) If $K(H_1) - K(H_2) = K(H)$, then $\Gamma(H_1) - \Gamma(H_2) = \Gamma(H)$.
(iii) If $K(H_1)K(H_2) = K(H)$, then $\gamma(H_1)\gamma(H_2) = \gamma(H)$.
(iv) The converses of (i), (ii), (iii).

All follow on comparing 5.00 to 5.10 with 5.11.

*5.14. Let now $x = p$, $z = p^{-s}$, where p denotes any positive prime number, and s is as yet arbitrary (unity is not counted as a prime); with the stipulation, that *as variables*, p, p^{-s} are *still independent;* e. g., if $h(x) = x$, $t = 1$, then $h(x)z^t$ is p/p^s, and this *is not further reducible*, viz., *is never* $1/p^{s-1}$. The product sign extending to all prime numbers p,

$$\text{(i)}\quad \Pi(1 + h_1(p)/p^{t_1s} + h_2(p)/p^{t_2s} + \cdots + h_r(p)/p^{t_rs})$$

is denoted by $[\gamma(H)]$, where $I(H) = |\,t\,|_r$, $B(H) = [h]_r$. If $[\gamma(H)]$ be expanded *formally*, the result will be denoted by $[\gamma(H)]\sim$ the formal expansion,[1] $\sim$ here being, *formally equivalent to.* The *evaluation* of the formal expansions is nowhere considered in this paper, so, strictly, questions of summability, etc., are irrelevant. However, it is presently shown that for the class of functions discussed [cf. 0.00 (ii)], s may always be so chosen that all products and series are absolutely convergent, and hence, $=$ may replace $\sim$. Formal multiplications, etc., are to be carried out as if the series and products were already proved to be absolutely convergent; in all cases the results have only a formal significance.

†5.15. With the notation of 5.14, $[\gamma(H)] \sim \sum_{n=1}^{\infty} H(n)/n^s$. The case $n = 1$ is provided for in 0.03; the rest is by direct comparison of 5.14 (i) and 3.14; cf. also 1.10, 3.22.

†5.16. ψ, ψ_1, ψ_2 being functional forms abstracted from numerical functions as defined in 0.03 (N. B. Not necessarily factorable), connected by the relation $\psi \sim \psi_1\psi_2$ [3.01], then $\sum_{n=1}^{\infty} \psi_1(n)/n^s \cdot \sum_{n=1}^{\infty} \psi_2(n)/n^s \sim \sum_{n=1}^{\infty} \psi(n)/n^s$. For, evidently the coefficient of $1/n^s$ in the formal product on the left is $\Sigma\psi_1(d)\psi_2(n/d)$, the summation being over all divisors d of n; this is $\psi(n)$.

[1] In this connection, the sign $\sim$ may be interpreted, *asymptotically equal to*, in the sense of Poincaré. [Cf. T. J. I'A. Bromwich, *Introduction to Infinite Series*, pp. 330–337.]

The theorem is valid if ψ, ψ_1, ψ_2 are all factorable, for factorable functions are included in those defined in 0.03.

†5.17. If $H \sim H_1H_2$, then $[\gamma(H_1)\gamma(H_2)] \sim [\gamma(H)]$. For, *formally*, $[\gamma(H_1)\gamma(H_2)]$ is $[\gamma(H_1)][\gamma(H_2)]$ on rearranging the factors; applying 5.15, 5.16, the result follows.

†5.18. If H is an algebraic primitive form [3.23] whose index [3.18] is of finite degree [4.00], a finite value of s may be assigned (in several ways) which renders both $[\gamma(H)]$ and $\sum_{n=1}^{\infty} H(n)/n^s$ absolutely convergent, and hence, for this value of s, [5.15], $[\gamma(H)] = \sum_{n=1}^{\infty} H(n)/n^s$.

For, referring to 5.11, 5.14; (i) $|\gamma(H)| \leqq 1 + \sum_{i=1}^{r} |h_i(p)/p^{t_is}|$; and each $h_i(p)$ is a polynomial (of finite degree) with integral coefficients, hence for $h_i(p)$ there is a positive integral constant (viz., independent of p), m_i, such that $|h_i(p)| \leqq m_ip^{g_i}$, where g_i is also constant. Hence, the right of (i) is $\leqq 1 + \sum_{i=1}^{r} |m_ip^{g_i}/p^{t_is}|$; (ii). But, $t_i > t_j$ when $i > j$, hence p^{t_rs} is the L. C. D. of the fractions in (ii), which $\leqq 1 + \left\{ \sum_{i=1}^{r} | p^{(t_r - t_i)s}m_ip^{g_i} | \right\} \Big/ p^{t_rs}$; (iii). All the m_i being positive integers, the numerator of (iii) is $\leqq mp^{t+g}$, where $m = m_1m_2 \cdots m_r$; $g = g_1 + g_2 + \cdots + g_r$; and $t = t_r^2 - (t_1 + t_2 + \cdots + t_r)$; the result being obtained by multiplication, etc. Hence, m, $t + g$ being positive integers, independent of p, $|\gamma(H)| \leqq 1 + mp^{t+g}/p^{t_rs}$; whence, if $s > \{\log m + (t + g) \log p\}/t_r \log p$, [which is, in effect, independent of p, for $\log m/\log p$, may, for the inequality be taken $= 0$], $mp^{t+g}/p^{t_rs} < 1$; hence, for s so chosen, $[\gamma(H)]$, and hence $\sum_{n=1}^{\infty} H(n)/n^s$, is absolutely convergent. Similarly, it is clear, that for $\epsilon \neq 0$, preassigned, s may be chosen so that, (iv) $mp^{t+g}/p^{t_rs} < |\epsilon|$; also, since m, $t + g$, t_r are positive integers, the left of (iv) approaches 0 as a limit as s approaches ∞. No attempt has been made to find a lower limit for s, as the actual values of the products, etc., are irrelevant to an arithmetical theory as outlined in 0.01.

†5.19. If for all values of s which are such that both series converge absolutely, $\sum_{n=1}^{\infty} H_1(n)/n^s = \sum_{n=2}^{\infty} H_2(n)/n^s$, then $H_1(n) = H_2(n)$; the H_1, H_2 being as in 5.18. Assume that for $n \leqq m$, $H_1(n) = H_2(n)$. Then, from the equality of the two series,

$$H_1(m + 1) + \sum_{n=2}^{\infty} H_1(m + n) \left(\frac{m + 1}{m + n}\right)^s = H_2(m + 1) + \sum_{n=2}^{\infty} H_2(m + n) \left(\frac{m + 1}{m + n}\right)^s.$$

In this, letting s become infinite, from 5.18, each term on either side, except the first, vanishes; hence $H_1(m + 1) = H_2(m + 1)$. But, by 0.03, $H_1(1) = H_2(1)$; hence the induction is complete.

5.20. The theorems in 5.18, 5.19 need not be explicitly used; all results *found* by their use, in relation to numerical functions, may be easily verified à posteriori; but their use is not only convenient, but essential to the rapid finding of new relations. But, it is emphasized that the introduction of an infinite process, which has no place in a pure arithmetical theory,[1] is not essential to the *proof* of any result in the present theory. In order to give independent proofs of the theorems, it is only necessary to substitute the actual sums, etc., in place of ideal products, etc., in the equivalences, and perform the necessary reductions by direct calculation, when, in each case, an identity results.

†5.21. If $H \sim H_1H_2$, and $c[\gamma(H_1)\gamma(H_2)] \sim [\gamma(H)]$, then $c = 1$. This is obvious but important in connection with units. Also, if $[\gamma(H_1)\gamma(H_2)\gamma(H_3)] \sim [\gamma(H)]$, with $H \sim H_1H_2$, then $\gamma(H_3)$ is 1, and may be represented by $\gamma(H_4)/\gamma(H_4)$, where H_4 is an arbitrary primitive form. But also, $\gamma(H_3)$ may be represented by $F(x)/F(x)$, where $F(x)$ is entirely arbitrary; this leads to the conception, important for the present purposes, of

*5.22. *Relevant Units.* The discussion being restricted to the class of all functions derived according to the several definitions, § 3, from *algebraic primitives* as defined in 3.23, those units, ϵ_i, which arise from $[\gamma(H_i)/\gamma(H_i)]$, and no others, wherein H_i is an algebraic primitive, are *relevant*, and *henceforth, unit, when used in connection with functions derived as described from algebraic primitives, shall signify relevant unit.* The method in which relevant units are generated is: let $[\gamma(H_i)] \sim \sum_{n=1}^{\infty} \psi_i'(n)/n^s$; and $[1/\gamma(H_i)] \sim \sum_{n=1}^{\infty} \psi_i''(n)/n^s$, then $[\gamma(H_i)/\gamma(H_i)] \sim \sum_{n=1}^{\infty} \psi(n)/n^s$, where $\psi \sim \psi'\psi''$; but clearly, the left is 1 for all values of s, identically; hence $\psi(n)$ is a unit function, and ψ is written ϵ_i; or $\psi_i'\psi_i'' \sim \epsilon_i$.

*5.23. *Positive and Negative Functions.* Let H_i $(i = 1, 2, \cdots)$ denote algebraic primitive forms; a_i $(i = 1, 2, \cdots)$ positive non-zero integers; ϵ's units. All expressions of the form $H_1^{a_1}H_2^{a_2}\cdots H_s^{a_s}$, say H, are *positive functions*, all expressions of the form ϵ/H (ideal, cf. 3.26) are *negative functions.* Also, if H' is a positive, H'' a negative function, $H'H''$ is a *mixed function;* in all:

The entire theory might have been constructed on positive functions, regarding negative and mixed functions as (ideal) fractions; this, to a certain extent, is done in connection with addition, but the method adopted seems better for 0.00 (ii) as the majority of the numerical functions in common use are mixed, and from this point of view it seems advisable to regard H and ϵ/H as distinct functions according to their *arithmetical definitions* [0.05], rather than to force all into the positive mould. Again, some of the negative functions, e. g., ν, [2.041 (v)], are highly important, whereas their reciprocals, (e. g., ϵ/ν), that is, the corresponding positives, are of little importance arithmetically; e. g., ϵ/ν does not seem to have arisen yet in the ordinary arithmetic.

[1] Cf. G. B. Mathews: *Theory of Numbers*, Part I, p. 1.

†5.24. (i) If the G. C. D. of $\gamma(H_1)$, $\gamma(H_2)$ is $\gamma(H)$, and if $\gamma(H_1) = \gamma(H)\gamma(H_1')$; $\gamma(H_2) = \gamma(H)\gamma(H_2')$, then $[\gamma(H_1)/\gamma(H_2)] \sim [\gamma(H_1')/\gamma(H_2')]$.

(ii) If s be chosen so that all the series are absolutely convergent, and if $\psi'\psi'' \sim \epsilon$, also $[\gamma(\psi')] \sim \sum_{n=1}^{\infty} \psi'(n)/n^s$ and $[\gamma(\psi'')] \sim \sum_{n=1}^{\infty} \psi''(n)/n^s$, then $\gamma(\psi')\gamma(\psi'') = 1$; hence, $[1/\gamma(\psi')] \sim \sum_{n=1}^{\infty} \psi''(n)/n^s$; viz., the generating function of the reciprocal (ideal) of ψ' is the reciprocal of the generating function of ψ'; for $[1/\gamma(\psi')]$ is formally $1/[\gamma(\psi')]$, and by hypothesis, here $=$ may replace $\sim$ (in connection with generating functions); hence,

(iii) H_i, F_j being algebraic primitive forms, a_i, b_j positive non-zero integers, $(i = 1, \cdots, r;\ j = 1, \cdots, k)$, the generating function of the mixed function $H_1^{a_1}H_2^{a_2}\cdots H_r^{a_r}/F_1^{b_1}F_2^{b_2}\cdots F_k^{b_k} \equiv H$, is

$$\gamma(H) \equiv \{\gamma(H_1)\}^{a_1}\{\gamma(H_2)\}^{a_2}\cdots\{\gamma(H_r)\}^{a_r}/\{\gamma(F_1)\}^{b_1}\{\gamma(F_2)\}^{b_2}\cdots\{\gamma(F_k)\}^{b_k}.$$

For, by repeated application of 5.17, the generating function of $H_i^{a_i}$ is $\{\gamma(H_i)\}^{a_i}$; whence the result follows immediately.

†5.25. From 5.22, 5.24 (or independently) it is easy to see, that if H is an algebraic prime primitive [3.36], $\gamma(H)$ is irreducible, and conversely. [This follows easily by reductio ad absurdum from 5.17, and the definitions in 3.36, referring also to 3.34 and 5.22.]

Thus, the further study of positive, negative and mixed functions is reduced to a comparison, so far as ideal multiplication is concerned, with the theory of polynomials in two variables, whose coefficients are integers, and *whose absolute term in each case is unity.*[1]

The reader may prefer to omit the rest of § 5, also § 6, and pass at once to the illustrations in §§ 7, 8, which deal chiefly with numerical functions in common use, and illustrate sufficiently the arithmetical character of the theory which is constructed in §§ 1–6. Questions of *ideal congruence,* also of *ideal forms,* will have to be left, for lack of space, until the second part. In all cases where infinite series are used, it will be assumed that s has been so chosen that the series converge absolutely [cf. 8.07].

5.251. Generators and generating functions have been defined; occasionally, in reference to $\psi(n)$, either term will be used to denote the series whose nth term is $\psi(n)/n^s$, and this will be denoted by $\Gamma\psi$, or $\gamma(s)$, $\cdots$, etc.; in no case will there be cause for confusion [used chiefly in the illustrations].

†5.26. If H is abstracted from a mixed function, then $\gamma(H)$ is (i) irreducible, viz., an algebraic fraction in its lowest terms, or (ii) reducible, and the G. C. D. of the numerator and denominator is the generating function of all latent units [3.35] contained as ideal factors in H. [5.24, and the several definitions, etc.]; similarly,

†5.27. If the generating function of either a positive or negative function

[1] The *Introduction to Higher Algebra,* of M. Bôcher [New York, 1907], Ch. XVI, may be consulted for the necessary theorems on reducibility, etc., in regard to polynomials, which henceforth will be assumed without further notice. The domain of reducibility is [1]; other domains enter when the (ideal) theory of forms in relation to mixed, etc., functions is considered.

be resolved into its irreducible factors, say

$$\gamma(H) \equiv f(x, z) = \{f_1(x, z)\}^{r_1}\{f_2(x, z)\}^{r_2}\cdots\{f_s(x, z)\}^{r_s},$$

where the r's are all either positive or negative integers, then each $f_i(x, z)$ is the generating function of a prime primitive, H_i, and $H \sim H_1^{r_1}H_2^{r_2}\cdots H_s^{r_s}$ is the (*obviously unique*), resolution of H into prime factors (ideal).

†5.28. H, H_1, H_2 being positive, negative or mixed functions, each of (i), (ii), (iii) is formally equivalent to[1] each of the other two (at once, from comparison of (i), (iii) with (ii) and 5.11);

$$\text{(i) } H \sim H_1H_2; \text{ (ii) } K(H) = K(H_1)K(H_2); \ \gamma(H) = \gamma(H_1)\gamma(H_2).$$

The like is obviously *false*, if the functions are not precisely as specified, viz., if they are of the more general kind in 0.03; for, in that case, there is no appeal to the theory of polynomials, etc.

†5.29. Every mixed function being of the form given in 5.24 (iii), by successive applications of 4.60, or, as a corollary from the theory of polynomials, it is easy to show that: (i) the assemblage of mixed functions is denumerable; whence, as a corollary, or independently; (ii) each of the assemblages of positive or negative functions is denumerable; and hence, that (iii) the assemblage of characteristics of algebraic primitives (all units being relevant), is denumerable.

5.30. Considering H, a positive or negative functional form, and $\gamma(H)$ as a function of z, [3.11], by 3.24, the extent is *finite*, viz., in $\gamma(H)$ there is but a finite number of terms. But, if all definitions, etc., so far, remain unchanged, except that the extent is *essentially infinite*, a new kind of numerical function, a *transcendental numerical function* is defined. E. g., if $\gamma(H) = 1 - z/2 + z^2/3 - z^4/4 + \cdots$, H is transcendental. At present, it is sufficient to distinguish;

*(i) *algebraic numerical functions;* those whose generators are algebraic functions of z;

*(ii) *transcendental numerical functions;* those whose generators are transcendental functions of z.

As (ii) is considered fully in the second part, it need only be pointed out, that, *as functions of their arguments*, what are here called *algebraic* numerical functions, are *transcendental* in the usual sense of the Theory of Functions; but (i), (ii) makes a rational distinction between certain classes of functions. There seems to be no example of (ii) at present in arithmetic (or in analysis); it corresponds to the case of an essentially infinite index, and hence, to the segregation of all integers into an infinite number of classes irreducible to a finite number of classes. The properties of functions (ii), are (as will be shown), *arithmetically* [0.01] identical with the analytical properties of transcendental functions (in the ordinary sense).

*5.31. If in 5.24, each primitive is *algebraic* in the sense of 5.30; viz., if

[1] In the sense of mathematical logic.

for each H_i, $\gamma(H_i)$ is an *algebraic function of z, x,* then the mixed function there defined is an *algebraic mixed function;* and, the totality of algebraic positives, negatives [5.30], mixeds [5.31], constitute the assemblage of *algebraic numerical functions;* briefly, *A-functions.*

†5.32. All H's denoting primitive A-functions: (i) $B(H_1)B(H_2) = B(H_1H_2)$; (ii) $K(H_1)K(H_2) = K(H_1H_2)$; (iii) these remain true if either or both H_1, H_2 be multiplied ideally by any units. (iv) If a value of $K(H_1)/K(H_2)$ exists, it is unique. (v) $K(H)$ may be written as a product $K(H_1)K(H_2)\cdots K(H_r)$ wherein $K(H_i)$ $(i = 1, \cdots, r)$ is further irresoluble into factors, in one way only, $K(HH')$ and $K(H)$ being considered identical if H' is a unit. Similarly, for $B(H)$.

The proofs are immediate consequences of the definitions and the corresponding theorems on polynomials. The (v) is false unless the primitives are A-functions.

*5.33. *Ideal Addition.* All H's being as in 5.32; referring to 5.13, (i), (ii), H is unique when H_1, H_2 are given, and, the *primitive A-function* [5.31] *whose characteristic is* $K(H_1) + K(H_2)$ is *the ideal sum of the primitive A-functions whose characteristics are $K(H_1)$, $K(H_2)$ respectively;* from 5.13 (ii), the *ideal difference of two primitive A-functions* is similarly defined.

It is clear that in no quantitative sense is an ideal sum or difference a sum or difference; the ideal sum expresses a relation between functions that is only remotely connected with their arguments.

†5.34. Ideal addition is associative and commutative; and, with respect to ideal multiplication, is distributive. Also (an important consequence for 0.00 (i) and (ii)), the property of factorability [0.04] is invariant under ideal addition and subtraction.

*5.35. There is another sepcies of addition, important for the sequel; it may be called *mixed addition,* and corresponds to the compound multiplication of 3.12. If ψ, ψ_1, ψ_2 are any functional forms, $\psi_1(n) + \psi_2(n)$ is *not* a factorable function, since, for $n = 1$, the value is 2, violating 0.04. But $\sum_{(n)}\psi(d)\{\psi_1(n/d) + \psi_2(n/d)\}$ [cf. 3.00], is evidently $\psi'(n) + \psi''(n)$, where $\psi' \sim \psi\psi_1$; $\psi'' \sim \psi\psi_2$; and hence, treating $\psi_1 + \psi_2$ as if it were a numerical function (as defined in 0.04), the ideal product of ψ and $\psi_1 + \psi_2$ is $\psi\psi_1 + \psi\psi_2$. Similarly for $\psi_1 + \psi_2 + \cdots + \psi_r$ $(r > 2)$; and, with this significance, $\psi_1 + \psi_2 + \cdots + \psi_r$ is a *mixed sum,* denoted by $|\,\psi_1 + \psi_2 + \cdots + \psi_r\,|$; and it has just been seen that

†5.36. (i) $\psi\,|\,\psi_1 + \psi_2 + \cdots + \psi_r\,| \sim |\,\psi\psi_1 + \psi\psi_2 + \cdots + \psi\psi_r\,|$; also, evidently, (ii) $|\,\psi_1 + \psi_2\,||\,\psi_1' + \psi_2'\,|$, the product being taken ideally, $\sim \psi_1\psi_1' + \psi_1\psi_2' + \psi_2\psi_1' + \psi_2\psi_2'$.

†5.37. If ψ_1, ψ_2 are factorable, $\psi_1 + \psi_2$ is in general not factorable. For, if $dv(n_1, n_2) = 1$, the necessary and sufficient condition that $\psi_1 + \psi_2$ be factorable, is, that for *all* pairs n_1, n_2 satisfying $dv(n_1, n_2) = 1$, shall (i): $|\,\psi_1(n_1) + \psi_2(n_1)\,||\,\psi_1(n_2) + \psi_2(n_2)\,| \sim |\,\psi_1(n_1n_2) + \psi_2(n_1n_2)\,|$; but, $\psi_1(n_1n_2) = \psi_1(n_1)\psi_2(n_2)$; and $\psi_2(n_1n_2) = \psi_2(n_1)\psi_2(n_2)$; whence, from (i), the necessary

and sufficient condition becomes, $|\psi_1(n_1)\psi_2(n_2) + \psi_2(n_1)\psi_1(n_2)| \sim 0$; which is not in general (if ever) satisfied.

5.38. The theorem in 5.37 is that which necessitated the introduction of ideal sums as defined; without ideal sums, a complete arithmetical theory of A-functions is impossible. Ideal addition is also the basis, in the second part, of the theories of congruences and forms for functions (ideal); it is also necessary to the more advanced parts of the theory, when, in order to complete 0.00 (i) an analogue for functions must be devised for DEDEKIND's theory of Ideals. This latter may, ab initio, be used as the groundwork, and similar theories to those of this part, be constructed, and finally, the whole of DEDEKIND's theory is placed in (1, 1) self-correspondence by means of the numerical functions proper to it. Again, for 0.00 (ii), the whole theory of characteristics, indices, etc., could have been omitted, and it is not to be inferred that by these means alone can 0.00 (i) be carried out in relation to 0.00 (ii); but, it is one of the simplest ways, and also, by choosing it, when taken in its most general form, a complete body of interconnected *theories*, each isomorphic to the other and to arithmetic in a manifold way, is obtained. Ideal addition and its consequences are only of secondary importance in relation to the integers; in regard to the *arithmetical properties of the functions as such*, it is of fundamental importance. It is easy to see, combining the results of this section, and comparing with 0.01;

†5.39. There is an arithmetical theory of characteristics of A-functions; moreover, in this theory, division is unique in the ordinary sense of the arithmetic of integers, and not dependent upon classes (as in the case of indices); similarly

†5.40. For functional sets, each set being the base of an A-function, there is an arithmetical theory; ideal addition for bases being defined thus:

$$B(H_1)+B(H_2)=B(H_1+H_2) \text{ when and only when } K(H_1)+K(H_2)=K(H_1+H_2).$$

*5.41. Referring to 5.24, the *divisors* of the mixed function there defined, *provided that each H, F is a prime primitive,* are defined by $H_1^{\alpha_1}H_2^{\alpha_2}\cdots H_r^{\alpha_r}/F_1^{\beta_1}F_2^{\beta_2}\cdots F_k^{\beta_k}$; where $0 \lesseqgtr \alpha_i \lesseqgtr a_i$ $(i = 1, \cdots, r)$, and $0 \lesseqgtr \beta_j \lesseqgtr b_j$ $(j = 1, \cdots, k)$: and, by convention, *all unit divisors*, ϵ, correspond to the case $\alpha_i = \beta_j = 0$ $(i = 1, \cdots, r;\ j = 1, \cdots, k)$.

If any H, F is *not a prime primitive*, the corresponding generating function is resolved, according to 5.25, into its irreducible factors, and then 5.24 as assumed in 5.41 results; whence, etc. Similarly, from 5.27, the *divisors of a positive or negative function* are defined; whence clearly,

†5.42. An A-function has only a finite number of distinct divisors; *distinct* functions (as divisors) being non-identical, these latter being defined in 3.34.

†5.43. (i) By means of §§0 to 5, it may be seen that there is an arithmetical theory for A-functions, which include most (if not all) of the numerical functions in common use. (ii) For convenience of comparison with the ordi-

nary arithmetic, some general theorems are placed together in § 6; in a few cases only is it necessary to give formal proofs.

§ 6. General Theorems.

6.00. *Prime* shall be as defined in 3.36; *unit* as in 5.22; *function*, when unqualified, *any one* of the kinds in 5.23. Cf. 5.43 (ii).

†6.01. The assemblages of (i) positive, (ii) negative, (iii) mixed, functions are denumerable. [At once from 5.24; (iii); 5.27; 5.25 by repeated applications of 4.60.]

†6.02. Every unit is a mixed function; no positive function, and no negative function, can be a unit.

†6.03. There is a denumerable infinity of units that are distinct arithmetically [0.05] from each other.

From 6.01, 6.02; the units are distinct arithmetically, for obviously, if H_1, H_2, $\cdots$ are either positive or negative functions which are distinct, then, the arithmetical definitions of H_1/H_1; H_2/H_2; $\cdots$ are distinct. [Examples in 7.09.]

†6.04. There is a denumerable infinity of positive, likewise of negative functions, that are prime.

That there is an infinity of such, follows from the known theorem[1] that if $\gamma(H_p) = (1 - z^p)/(1 - z)$, where p is a prime integer, $\gamma(H_p)$ is irreducible; applying 5.25; 6.01, the theorem follows; hence,

†6.05. The assemblage of primes is denumerable.

†6.06. The assemblage of functions is denumerable. [Combining 6.01 (i) to (iii).]

†6.07. As ideal multipliers, the units are identical.

†6.08. If the absolute unit be taken as the identity element, so that ϵ and $1/\epsilon$ are ideal reciprocals, the units form a group with respect to ideal multiplication.

†6.09. All units are identically equivalent.

†6.10. The numerical values for any general argument of any function and all its associates [3.32] are equal.

†6.11. Any function may be resolved into its prime (function) factors in one way only. [Numerous examples in § 7.] [Directly from 5.21 to 5.26; paying attention to 3.34.] Cf. 00.1 (2).

†6.12. If a prime function is a factor of a product of functions, it must divide at least one of the factors (the product and division being both ideal). Similarly to 6.11.

†6.13. Any function is ideally divisible by a finite number only of distinct primes, and has only a finite number of ideal divisors [5.42].

†6.14. If a prime divides neither of two functions, it cannot divide their product. [N. B. Cf. 5.23; remarks; otherwise the theorem has no meaning; also 5.41.] Similarly to 6.11; and

[1] Cf. G. B. Mathews: l. c., p. 186.

†6.15. The prime factors of a positive or negative function are respectively all positive or negative functions; those of a mixed function may be either.

†6.16. Neither a positive nor a negative function can contain latent units; a mixed function may, but not necessarily.

†6.17. There is an arithmetical theory for functions. [By combining the theorems of this § and of § 5.] In this theory, there is an infinity of units.

†6.18. There is an arithmetical theory for positive functions; in this theory there is but a single unit, the absolute unit, 1.

†6.19. There is *not* an arithmetical theory for negative functions. [The break comes in the lack of an ideal addition which shall make the ideal sum of two negatives a negative.]

†6.20. With respect to ideal multiplication the functions form a group.

†6.21. With respect to ideal addition the positive functions form a group, the identity element being, by convention, 0. [The relations of ideal addition to *all* the functions are not considered until the second part; they correspond to the *rational* numbers.]

†6.22. With respect to ideal multiplication, neither the positive nor the negative functions form a group; but each forms a semi-group.[1]

†6.23. With respect to algebraic addition and ideal multiplication, mixed sums [5.35] form a group; 0 and ϵ being the respective identity elements; and

†6.24. In similar respects, each of, positive, negative, functions form a semi-group.

†6.25. There is a denumerable infinity of functions which are associates of (or which are identically equivalent to) a given mixed function.

†6.26. The abstract operational laws of ordinary algebra are valid when the operands are functional forms and $+$ is algebraic (giving mixed sums), $\times$ ideal; also when $+$, $\times$ are both ideal, provided, however, that in this case[2] the functional forms be abstracted from only positive functions [from § 5; replacing the functions by their generatiors, applying the corresponding theorems on polynomials, and re-translating to functions, etc.], whence

†6.27. Any algebraic identity in which occur only sums and products, or either alone, and hence also powers whose exponents are positive integers, is true when sums are interpreted as mixed, products and powers being ideal.

†6.28. There is no analogue in functions to a complete system of residues for a modulus,[2] if only mixed sums occur in the theory, but, among many others;

†6.29. The sums being mixed, powers ideal, n a prime integer, the ψ_i $(i = 1, \cdots, r)$ functions as defined in 0.03,

$$(\psi_1 + \psi_2 + \cdots + \psi_r)^n \equiv \psi_1^n + \psi_2^n + \cdots + \psi_r^n \bmod n;$$

[1] As this term does not seem to be widely used, it is defined: A set of elements form a semi-group if they have the *group property*, etc.

[2] Restriction removed in the second part, when all theorems of this § are made general for a field in which all four elementary operations are ideal, or in which 1, 2, or 3 are ideal, the rest ordinary.

viz., regarding ψ_i^n as a new function, distinct from ψ_i, similarly $\psi_1 + \cdots + \psi_r$ a function, which *qua function* is independent of ψ_i $(i = 1, \cdots, r)$, then for every value of the argument of these new functions, the foregoing congruence holds. [Similarly to 6.26.]

†6.30. If only mixed sums are admitted, there is no analogue in the theory to FERMAT's theorem.[1] For, by 6.28 there is no concept of a complete system of residues. In the case of positive functions, such may be readily imagined in connection with 5.33.

*†6.31. If a, b are relatively prime, ψ_1, ψ_2 functions which have no ideal factor common, $\psi_1^a \sim \psi_2^b$ introduces the concept of *irrational functions;* it will be shown that their properties are wholly analogous to those of irrational numbers, when these $\psi_1, \psi_2, \cdots$ are compounded according to ideal addition, multiplication, etc. There are no examples of these at present in arithmetic. Their extents are all infinite, and they are distinct from the functions in 5.30 (ii).

6.32. This list may be indefinitely extended; enough has been given to show the arithmetical character of the theory which has been constructed for the functions considered. It has not been thought necessary to write out formal proofs, as all are entirely elementary, and indeed, the theorems are direct consequences of §§ 3 to 5; details may, if desired, be supplied as suggested in 6.26. But the following are of an entirely different nature, and indicate that the theory already constructed is but the first in an infinite chain of theories, all abstractly identical to each other and to arithmetic, and in a sense which may be easily imagined, each link in the chain implies all that precede it. They differ from one another only in respect of the elements upon which they are constructed; also, each is directly applicable to the integers, but, as the chain is descended, the properties of the appropriate functions in each link, become more and more complicated, until, in most cases, even at the second link, they may safely be said to defy verbal definition, although *as functions*, their properties are as simple as in 6.00 to 6.30.

†6.33. The elements in the theory up to 6.31, are the functions which have grown naturally out of the unique factorization law (0.01) in ordinary arithmetic, the *arguments of the functions being integers;* call these the $\psi^{(1)}$ functions. Starting from 6.11, and the $\psi^{(1)}$ functions as arguments, §§ 1, 3 (especially 3.14) may be rewritten, replacing therein *number* by *function*, *prime number* by *prime function*. In this way are defined *primitive* $\psi^{(2)}$ *functions*, *prime* $\psi^{(2)}$ *functions*, and §§ 4 to 6, down to 6.31 are rewritten, a $\psi^{(2)}$ function being the result of replacing in a $\psi^{(1)}$ each prime number by a prime $\psi^{(1)}$ function, etc., as will be indicated on rewriting §§ 1, 3. [Thus, e. g., if $\psi^{(1)}(n) \equiv \varphi(n)$, then, if $\psi \sim \psi_1^{a_1}\psi_2^{a_2} \cdots \psi_r^{a_r}$ (ideal product), the corresponding $\psi^{(2)}(n)$ is $\psi(1 - 1/\psi_1) \cdots (1 - 1/\psi_r)(n)$, wherein the product is to be distributed, simplified as far as possible algebraically (into a mixed sum or difference of ideal products, etc.), and each function in the result is to have the argument n.] In particular, there will be a theorem 6.11 for $\psi^{(2)}$ functions; and there will

exist prime $\psi^{(2)}$ functions, upon which, and the new 6.11, the process may be repeated, giving $\psi^{(3)}$ functions (whose arguments are $\psi^{(2)}$ functions), and so on. Hence any theorem regarding $\psi^{(1)}$ functions may be expanded (rather, *unfolded*) into an infinite sequence of theorems. [Examples of this 6.33 are not considered until the second part, where also the inter-relations of the $\psi^{(1)}$, $\psi^{(2)}$, $\cdots$, are considered.]

6.34. This indicates one direction in which the theories of this paper may be extended. Another consists in basing all upon the theory of Ideals of DEDEKIND, instead of upon the integers; and others, more fundamentally distinct from the present theory than any of these, will also be given in their proper place. Another, simpler than any of these, arises as follows:

†6.34. The theorems of § 6 (to 6.33) may be shown to be valid for functions based upon the resolution of an integer into *simple* (rather than *prime*) factors [§ 1], by replacing n by n' [1.03], p_i by P_i [1.01], and 5.14 (i) by [cf. 1.10],

$$\Pi(1 + L(P)h_1(P)/P^{t_1 s} + \cdots + L(P)h_r(P)/P^{t_r s});$$

the Π extending over *all* simple numbers, unity excluded; together with a few simple and obvious corresponding changes elsewhere. Or, from another point of view, the result is a *direct* consequence of 6.11.

In the illustrations, it is a simple exercise to change each theorem into its correspondent upon this basis, and with a little care, the results may be translated in accordance with § 1.

†6.35. Finally, at any stage in 6.33, the prime or simple resolution may be adopted. Thus, up to the functions $\psi^{(r)}$ there are 2^r theories constructed. These differ in that, when finally *unfolded* down to $\psi^{(1)}$ functions, the properties of integers which are relevant to each of the 2^r theories, are distinct. But, each of the 2^r is of the same kind as that constructed for the $\psi^{(1)}$ functions, *abstractly*.

†6.36. The following special cases of 6.27 are useful; (i) if $\psi \sim \psi_1\psi_2$, then $\psi_1 \sim \psi_2'\psi$ where $\psi_2'\psi_2 \sim \epsilon$. Proving independently of 6.27, $\gamma(\psi) = \gamma(\psi_1)\gamma(\psi_2)$, from the given condition [5.28]; also $\gamma(\psi_2') = 1/\gamma(\psi_2)$ [5.24 (ii)], and, $\gamma(\psi_1) = \gamma(\psi)/\gamma(\psi_2) = \gamma(\psi)\gamma(\psi_2')$, whence from 5.28, the result follows. (ii) In the same way, if $\psi_1 \sim \psi_2$, and $\psi_3 \sim \psi_4$, then $\psi_1\psi_3 \sim \psi_2\psi_4$; $\psi_1\psi_4 \sim \psi_2\psi_3$; $\psi_1/\psi_3 \sim \psi_2/\psi_4$; etc.

†6.37. Upon sets as a basis, instead of upon integers, the $\psi^{(1)}$ functions [6.33] may be constructed, and the arithmetical theory of these is developed, by a few changes in the meanings of the terms, parallel to that for $\psi^{(1)}$ functions. Also, on these *set-numerical* functions, the processes of 6.33; 6.35 may be carried out. The interpretation of the results is made in accordance with 4.59.

6.38. A further generalization may be noted, although its nature will not be apparent until after 8.09 to 8.23. It consists in replacing the *integers* τ as well as the t, of 3.14 by functional symbols ψ, and repeating 6.33 to 6.35. There result properties of the functions in relation to their arguments that

are of an order entirely different to any so-far considered. It will also be proved that *any* $\psi^{(r)}$ function [6.35] is a definite $\psi^{(1)}$ function (but not conversely); and that the $\psi^{(1)}$ functions may be distributed into classes of $\psi^{(r)}$ $(r = 2, 3, \cdots)$ functions in essentially only one way. In the $\psi^{(2)}$ functions [e. g., in 6.33], 1 is the symbol of any relevant unit. In the $\psi^{(r)}$ functions, the relevant units $\epsilon^{(r)}$ play the part of unity in the arithmetical definitions of $\psi^{(1)}$ functions.

§ 7. Special Theorems and Illustrations.

7.00. These are chiefly upon the consequences of ideal multiplication; much more general theorems are given subsequently, also, other general methods of deriving and proving such relations will be considered. The most of those in this section are from Liouville's four articles,[1] wherein they are stated. From the present point of view, there are two problems in connection with any given function: (i) the resolution into prime factors [6.11]; (ii) the relation to other numerical functions. Of (i) a single detailed example will suffice [7.01]; the similar results in other cases, only are stated; and (ii) may be systematically accomplished, either from (i) by 6.26, 6.27, or directly from the generators; both methods are useful, and each suggests new functions and relations in great profusion. With but little practice, the direct reading of the arithmetical definition [0.05] from the generating function (or generator), and vice-versa, become very simple matters; thus, e. g., in 7.05; 7.07, all results may be derived from first principles as in 7.01, from the arithmetical definitions; once they are stated however, they become obvious at a glance. The example in 7.01 is more complicated than any in 7.05 or 7.07. For definitions of symbols, cf. 2.041; § 5; 3.12.

†7.01. To resolve $H(n) \equiv \varphi(D_2(n))\nu(n/D_2^{(2)}(n))$ into its prime factors. Clearly, H is factorable [0.04]. The finding of $\gamma(H)$ is reduced, therefore, to the summation and reduction of $1 + \sum_{\alpha=1}^{\infty} H(p^\alpha)z^\alpha$; where $z = p^{-s}$ [5.24]. But, $H(p) = 2$;

$$H(p^{2\alpha}) = \varphi(p^\alpha)\nu(p^{2\alpha}/p^{2\alpha}) = \varphi(p^\alpha)\nu(1) = \varphi(p^\alpha) = p^\alpha - p^{\alpha-1}.$$

And,

$$H(p^{2\alpha+1}) = \varphi(p^\alpha)\nu(p^{2\alpha+1}/p^{2\alpha}) = \varphi(p^\alpha)\nu(p) = 2\varphi(p^\alpha) = 2(p^\alpha - p^{\alpha-1}).$$

Hence,

$$\gamma(H) = \{1 + \varphi(p)z^2 + \cdots + \varphi(p^\alpha)z^{2\alpha} + \cdots\} \\ + 2\{z + \varphi(p)z^3 + \cdots + \varphi(p^\alpha)z^{2\alpha+1} + \cdots\}.$$

Whence, substituting for $\varphi(\ )$ its value, and writing

$$1 + \sum_{\alpha=1}^{\infty} p^\alpha z^{2\alpha} \equiv S = 1/(1 - pz^2);$$

[1] *Journal des Math.*, 2 Sér. (1857), t. 2. *Sur quelques fonctions numeriques*. The theorems from Liouville are in 7.10 and a few in 7.12.

$\gamma(H) = (S - z^2S) + 2(zS - z^3S) = (1 + 2z)(1 + z)(1 - z)/(1 - pz^2);$
viz. [cf. 5.24],

$$[\gamma(H)] = \sum_{n=1}^{\infty} H(n)/n^s.$$

Let

$$\gamma(H_1) = (1 + 2z);\ \gamma(H_2) = (1 + z);\ \gamma(H_3) = (1 - z);\ \gamma(H_4) = 1/(1 - pz^2).$$

Then, the required resolution is $H \sim H_1H_2H_3H_4$: since each generator is irreducible, and there are no latent units [3.35]; and by comparing with 2.04, 2.041, directly, $H \sim \mu \mid \mu^2 \mid \mid \mu^2\nu \mid \mid k_2u_{1/2} \mid$; thus H is the ideal product of the four primes [3.36], μ, $\mid \mu^2 \mid$, $\mid \mu^2\nu \mid$, $\mid k_2u_{1/2} \mid$ [5.27].

It is to be noted that $\mid \mu\mu \mid$ is written $\mid \mu^2 \mid$; $\mid\mid \mu^2 \mid \nu \mid\mid$ as $\mid \mu^2\nu \mid$; etc.; viz., μ^2, in $\mid \mu^2\nu \mid$, is *not* an ideal square; so in all cases. As all the functions used in § 7 are specializations of Ψ [§ 2], this will now be examined in some detail; powers, ψ^r, $\mid \psi_1\psi_2 \mid^s$, of functions are ideal, and ϵ is any unit, $\epsilon_1, \epsilon_2, \cdots$ distinct units [6.03]; again, cf. 3.12 and 2.04, 2.041 for the meanings of the symbols.

†7.02. Writing $\Psi\ (n;\ a, b, c, l) \equiv \Psi_l$, for $l = \pm 1$; and comparing with 2.01; 3.25; 3.26:

(i) $\gamma(\Psi_l) = (1 + cp^az^b)^l$ (ii) $\gamma(\Psi_l{}^m) = (1 + cp^az^b)^{lm}$; m any integer.

Hence [cf. 2.02],

†7.03. $\gamma(\psi) = (1 - p^az^b)$: $\gamma(\chi) = (1 + p^az^b)$: $\gamma(\zeta) = (1 + az^b)$.
$\gamma(\psi') = 1/\gamma(\psi)$: $\gamma(\chi') = 1/\gamma(\chi)$: $\gamma(\zeta') = 1/\gamma(\zeta)$.

†7.04. $\psi\psi' \sim \chi\chi' \sim \zeta\zeta' \sim \epsilon \sim \Psi_1{}^m\Psi_{-1}{}^m$ [7.03, 5.22, 5.24].

†7.05.

(1) $\gamma(\mu) = (1 - z)$. (2) $\gamma(k_r) = 1/(1 - z^r)$.
(3) $\gamma(u_r) = 1/(1 - p^rz)$. (4) $\gamma(\varpi) = 1/(1 + z)$.
(5) $\gamma(\mid \mu^2 \mid) = (1 + z)$. (6) $\gamma(\mid \varpi u_1 \mid) = 1/(1 + pz)$.
(7) $\gamma(\mid k_2u_{1/2} \mid) = 1/(1 - pz^2)$. (8) $\gamma(\mid \mu^2\nu \mid) = (1 + 2z)$.
(9) $\gamma(\mid \mu_{1/2}k_2u_{1/2} \mid) = (1 - pz^2)$. (10) $\gamma(\mid u_1\mu^2 \mid) = (1 + pz)$.
(11) $\gamma(\mid \mu^2(2^r - 1)^\nu \mid) = (1 + (2^r - 1)z)$. (12) $\gamma(\mid \mu u_r \mid) = (1 - p^rz)$.
(13) $\gamma(\mid \varpi u_r \mid) = 1/(1 + p^rz)$. (14) $\gamma(\mid \mu^2u_r \mid) = (1 + p^rz)$.

†7.06. The fourteen functions whose generators are given in 7.05, are all primes [5.27], except (2).

†7.07.

(1) $\gamma(\nu) = 1/(1 - z)^2$. (2) $\gamma(\varphi) = (1 - z)/(1 - pz)$.
(3) $\gamma(\sigma) = 1/(1 - z)(1 - pz)$. (4) $\gamma(\theta) = (1 + z)/(1 - z)$.
(5) $\gamma(\mid \varpi\theta \mid) = (1 - z)/(1 + z)$. (6) $\gamma(\mid u_1\nu \mid) = 1/(1 - pz)^2$.
(7) $\gamma(\mid \nu^2 \mid) = (1 + z)/(1 - z)^3$.
(8) $\alpha_1{}^{(r)}(n) \equiv \nu(n^r)$; $\gamma(\alpha_1) = (1 + (r - 1)z)/(1 - z)^2$.
(9) $\alpha_2{}^{(r)}(n) \equiv \nu(n)\nu(n^r)$; $\gamma(\alpha_2) = \gamma(\mid \nu\alpha_1{}^{(r)} \mid) = (1 + (2r - 1)z)/(1 - z)^3$.
(10) $\beta_1(n) \equiv \nu(n/D_2{}^{(2)}(n))$; $\gamma(\beta_1) = (1 + 2z)/(1 + z)(1 - z)$.

(11) $\beta_2(n) \equiv \varphi(D_2^{(2)}(n))$; $\gamma(\beta_2) = (1 - pz^2)(1 + z)/(1 + pz)(1 - pz)$.
(12) $\beta_3(n) \equiv n/D_2^{(2)}(n)$; $\gamma(\beta_3) = (1 + pz)/(1 + z)(1 - z)$.
(13) $\beta_4(n) \equiv \varphi(D_2(n))$; $\gamma(\beta_4) = (1 + z)^2(1 - z)/(1 - pz^2)$.
(14) $\beta_5(n) \equiv \varphi(D_2(n))\nu(n/D_2^{(2)}(n))$; $\gamma(\beta_5) = (1 + 2z)(1 + z)(1 - z)/(1 - pz^2)$.
(15) $\beta_6(n) \equiv D_2^{(2)}(n)$; $\gamma(\beta_6) = (1 + z)/(1 + pz)(1 - pz)$.
(16) $\beta_7(n) \equiv D_2(n)$; $\gamma(\beta_7) = (1 + z)/(1 - pz^2)$.
(17) $\beta_8(n) \equiv \nu(n/D_2^{(2)}(n))$; $\gamma(\beta_8) = (1 + 2z)/(1 + z)(1 - z)$.
(18) $\beta_9(n) \equiv \theta(D_2(n))$; $\gamma(\beta_9) = (1 + z^2)/(1 - z)$; $\gamma(\zeta(1, 2)) = (1 + z^2)$; $\zeta(1, 2)$ written for $\zeta(n; 1, 2)$.
(19) $\beta_{10}(n) \equiv \theta(n/D_2(n))$; $\gamma(\beta_{10}) = (1 + z)/(1 - z)$; or, $\beta_{10} \simeq \theta$.
(20) $\beta_{11}(n) \equiv \theta(n/D_2^{(2)}(n))$; $\gamma(\beta_{11}) = (1 + 2z)/(1 + z)(1 - z)$; or, $\beta_{11} \simeq \beta_8$.
(21) $\beta_{12}(n) \equiv \varpi(n/D_2^{(2)}(n))$; $\beta_{12} \simeq \varpi$.
(22) $\gamma(|\, \varpi\sigma \,|) = 1/(1 + z)(1 + pz)$. (23) $\gamma(|\, u_1\sigma \,|) = 1/(1 - pz)(1 - p^2z)$.
(24) $\gamma(|\, \theta^r \,|) = (1 + (2^r - 1)z)/(1 - z)$.
(25) $\gamma(|\, \varpi\alpha_1^{(r)} \,|) = (1 + (r - 1)z)/(1 + z)^2$.
(26) $\zeta_r \equiv u_0u_r$; $\gamma(\zeta_r) = 1/(1 - z)(1 - p^rz)$. $\zeta_r(n) \equiv$ the sum of the rth powers of the divisors of n; obviously, $\zeta_0 \simeq \nu$.
(27) $\gamma(|\, \nu\varpi \,|) = 1/(1 + z)^2$. (28) $\gamma(|\, u_r\nu \,|) = 1/(1 - p^rz)^2$.
(29) $\gamma(|\, u_r\zeta_r \,|) = 1/(1 + p^rz)(1 - p^rz)^2$.
(30) $\gamma(|\, u_r\zeta_{sr} \,|) = 1/(1 - p^rz)(1 - p^{(s+1)r}z)$.
(31) $\gamma(|\, u_r\zeta_s \,|) = 1/(1 - p^{r+s}z)(1 - p^rz)$.
(32) $\gamma(|\, \nu\alpha_1^{(m)} \,|) = (1 + (2m - 1)z)/(1 - z)^3$.

These all are modifications of 7.03, which comes directly from 7.02; thus, all the functions of 7.07, as well as all those of 7.05, arise from Ψ [7.02], which obviously gives ∞ more, primes and composites. Enough have been written down to illustrate the richness of this single primitive, the simplest of all. From 7.05; 7.07 there are the following resolutions into prime factors:

†7.08. (1) $\nu \sim u_0^2$; (2) $\varphi \sim \mu u_1$; (3) $\sigma \sim u_0u_1$; (4) $\theta \sim |\, \mu^2 \,|\, u_0$; (5) $|\, \varpi\theta \,| \sim \mu\varpi$; (6) $|\, u_1\nu \,| \sim u_1^2$; (7) $|\, \nu^2 \,| \sim |\, \mu^2 \,|\, u_0^3$; (8) $\alpha_1^{(r)} \sim |\, \mu^2(r - 1)^\nu \,|\, u_0^2$; (9) $\alpha_2^{(r)} \sim |\, \mu^2(2r - 1)^\nu \,|\, u_0^3$; (10) $\beta_1 \sim |\, \mu^2\nu \,|\, \varpi u_0$; (11) $\beta_2 \sim u_1 \,|\, \varpi u_1 \,||\, \mu_{1/2}k_2u_{1/2} \,|\, |\, \mu^2 \,|$; (12) $\beta_3 \sim \varpi u_0 \,|\, u_1\mu^2 \,|$; (13) $\beta_4 \sim |\, \mu^2 \,|^2 \,|\, k_2u_{1/2} \,|\, \mu$; (14) β_5; cf. 7.01; (15) $\beta_6 \simeq D_2^{(2)} \sim |\, \mu^2 \,|\, |\, \varpi u_1 \,|\, u_1$; (16) $\beta_7 \simeq D_2 \sim |\, \mu^2 \,|\, |\, k_2u_{1/2} \,|$; (17) $\beta_8 \sim |\, \mu^2\nu \,|\, \varpi u_0$; (18) $\beta_9 \sim \zeta(1, 2)u_0$; (19) $\beta_{10} \simeq \theta$; cf. 7.08 (4); (20) $\beta_{11} \simeq \beta_8$; (21) $\beta_{12} \simeq \varpi$; (22) $|\, \varpi\sigma \,| \sim \varpi \,|\, \varpi u_1 \,|$; (23) $|\, u_1\sigma \,| \sim u_1u_2$; (24) $|\, \theta^r \,| \sim |\, \mu^2(2^r - 1)^\nu \,|\, u_0$; (25) $|\, \varpi\alpha_1^{(r)} \,| \sim \zeta(r - 1, 1)\varpi^2$ [cf. 7.07 (18)]; (26) cf. 7.07 (26); (27) $|\, \nu\varpi \,| \sim \varpi^2$; (28) $|\, u_r\nu \,| \sim u_r^2$; (29) $|\, u_r\zeta_r \,| \sim |\, \varpi u_r \,|\, u_r^2$; (30) $|\, u_r\zeta_{sr} \,| \sim u_ru_{(s+1)r}$; (31) $|\, u_r\zeta_s \,| \sim u_{r+s}u_r$; (32) $|\, \nu\alpha_1^{(m)} \,| \sim \zeta(2m - 1, 1)u_0^3$; cf. 7.07 (18).

From 7.05; 7.07, are written down the following units; the (ideal) product of any number of units being again a unit, these may be multiplied together, some or all, to furnish new units; those written are of use in subsequent reductions by expression [3.35].

†7.09. (1) μu_0; (2) μk_1; (3) $u_r \,|\, \mu u_r \,|$; (4) $\varpi \,|\, \mu^2 \,|$; (5) $\mu^2\nu$; (6) $|\, \mu u_1 \,|\, \varphi u_0$; (7) $|\, \mu u_1 \,|\, \mu\sigma$; (8) $\varpi\mu\theta$; (9) $|\, \varpi u_1 \,|\, |\, u_1\mu^2 \,|$; (10) $|\, k_2\mu_{1/2} \,|\, |\, \mu_{1/2}k_2u_{1/2} \,|$; (11) $|\, \mu u_r \,|\, u_r$;

(12) $| \varpi u_r | | \mu^2 u_r |$; (13) $| \nu^2 | \varpi\mu^3$; (14) $\beta_2 | k_2 u_{1/2} | | u_1\mu^2 | | u_1\mu | \varpi$; (15) $\beta_3 | \varpi u_1 | | \mu^2 | \mu$; (16) $\beta_4 | \mu_{1/2}k_2u_{1/2} | \varpi^2 u_0$; (17) $\beta_6 | u_1\mu^2 | | \mu u_1 | \varpi$; (18) $\beta_7 | \mu_{1/2}k_2u_{1/2} | \varpi$; (19) $| \varpi\sigma | | \mu^2 u_1 | | \mu^2 |$; (20) $| u_1\sigma | |\mu u_1| |\mu u_2|$; (21) $\zeta_r\mu | \mu u_r |$; (22) $| \nu\varpi | | \mu^2 |^2$; (23) $| u_r\nu | | \mu u_r |^2$; (24) $| u_r\zeta_r | |\mu^2 u_r| |\mu u_r|^2$; (25) $| u_r\zeta_{sr} | | \mu u_r | | \mu u_{(s+1)r} |$; (26) $| u_r\zeta_s | | \mu u_{r+s} | | \mu u_r |$.

Each factor in these units is a prime, or the power of a prime; hence, in an exact sense these are the resolutions into prime factors of the units [this term does not strictly come under previous definitions]; and obviously, each unit is a mixed function. The cumbersome $| \mu_{1/2}k_2u_{1/2} |$ *will be denoted by* τ; the arithmetical definition is evident on referring to 7.05 (9); 7.03; 2.05 (i). In the following, the first and last equivalence of each chain [e. g., in $\psi_1 \sim \psi_2 \sim \cdots \sim \psi_r$ this is $\psi_1 \sim \psi_r$], is a theorem stated by LIOUVILLE; the intermediate links come from 7.05 to 7.09 by obvious substitutions, etc.; each link [e. g., $\psi_1 \sim \psi_2$; $\psi_2 \sim \psi_3$; $\cdots$; $\psi_1 \sim \psi_3$; $\cdots$, etc.] is clearly a theorem of the same kind. The numbering of the theorems 1.1, 1.2, $\cdots$, 2.7, $\cdots$, etc., means, e. g., 3.9, the 9th theorem in the third article of LIOUVILLE; the omitted numbers are considered together in 7.121. Also, the number of links in any chain may be continued indefinitely by multiplication by units; here, the aim has been to reduce the links as far as possible consistently with clearness; ϵ's are units.

7.10.

(1.1) $u_0\varphi \sim u_0\mu u_1 \sim u_1$ [GAUSS, *D. A.*, § 39]. (1.2) $u_0 | u_1\sigma | \sim u_0u_1u_2 \sim u_2\sigma$.
(1.3) $u_0\sigma \sim u_0^2u_1 \sim \nu u_1$. (1.4) $\varphi\nu \sim \mu u_1 u_0^2 \sim u_0u_1 \sim \sigma$.
(1.5) $\theta\nu \sim | \mu^2 | u_0^3 \sim | \nu^2 |$. (1.6) $\sigma^2 \sim u_0^2u_1^2 \sim \nu | u_1\nu |$.
(1.7) $\nu^2 \sim u_0^4 \sim u_0^4\varpi | \mu^2 | \sim u_0\varpi | \mu^2 | u_0^3 \sim k_2 | \nu^2 |$.
(2.1) $u_0\alpha_1^{(2r)} \sim u_0 | \mu^2(2r-1)^\nu | u_0^2 \sim \alpha_2^{(r)}$.
(2.2) $\alpha_1^{(2r)}\nu \sim | \mu^2(2r-1)^\nu | u_0^4 \sim u_0\alpha_2^{(r)}$.
(2.3) $u_0\alpha_1^{(2)} \sim u_0 | \mu^2 | u_0^2 \sim u_0^3 | \mu^2 | \sim | \nu^2 |$. (2.4) $\theta\nu \sim | \mu^2 | u_0^3 \sim | \nu^2 |$.
(2.5) $\theta\nu \sim u_0\alpha_1^{(2)}$ [2.3; 2.4]. (2.6) $\varphi\sigma \sim \mu u_1 u_0 u_1 \sim u_1^2 \sim | u_1\nu |$.
(2.7) $u_1\sigma \sim u_0u_1^2 \sim u_0 | u_1\nu |$. (2.8) $u_0\varpi \sim k_2$.
(2.9) $| \varpi\theta | \nu \sim \mu\varpi u_0^2 \sim u_0\varpi \sim k_2$. (2.10) $| \varpi\theta | \alpha_1^{(2)} \sim \mu\varpi | \mu^2 | u_0^2 \sim \mu u_0^2 \sim u_0$.
(2.11) $u_0 | \varpi\theta | \sim u_0\mu\varpi \sim \varpi$. (2.12) $\varpi\theta \sim \varpi | \mu^2 | u_0 \sim u_0$.
(2.13) $| \varpi\theta | \theta \sim \mu\varpi\theta \sim \mu\varpi | \mu^2 | u_0 \sim \mu u_0\varpi | \mu^2 | \sim \epsilon_1\epsilon_2 \sim \epsilon$.
(2.14) $\varpi\nu \sim \varpi u_0^2 \sim u_0k_2$. (2.15) $\varpi\sigma \sim \varpi u_0u_1 \sim k_2u_1$.
(3.1) $\alpha_1^{(2)}\varphi \sim | \mu^2 | u_0^2\mu u_1 \sim | \mu^2 | u_0u_1 \sim u_1\theta$.
(3.2) $u_1\alpha_1^{(2)} \sim u_1 | \mu^2 | u_0^2 \sim | \mu^2 | u_0u_0u_1 \sim \theta\sigma$. (3.3) $\alpha_1^{(2)}\varpi \sim | \mu^2 | u_0^2\varpi \sim u_0^2 \sim \nu$.
(3.4) $| \nu^2 | | \varpi\theta | \sim | \mu^2 | u_0^3\mu\varpi \sim | \mu^2 | \varpi u_0\mu u_0^2 \sim u_0^2$.
(3.5) $| \nu^2 | | \varpi\theta | \sim | \mu^2 | u_0^3\mu\varpi \sim | \mu^2 | u_0^2\varpi \sim \alpha_1^{(2)}\varpi$.
(3.6) $\varphi\alpha_2^{(r)} \sim \mu u_1 | \mu^2(2r-1)^\nu | u_0^3 \sim u_1\alpha_1^{(2r)}$.
(3.7) $\theta\alpha_2^{(r)} \sim u_0 | \mu^2 | | \mu^2(2r-1)^\nu | u_0^3 \sim u_0^2 | \mu^2 | | \mu^2(2r-1)^\nu | u_0^2 \sim \alpha_1^{(2)}\alpha_1^{(2r)}$.
(3.8) $\alpha_1^{(2r)}\sigma \sim | \mu^2(2r-1)^\nu | u_0^2u_0u_1 \sim u_1\alpha_2^{(r)}$. For omitted numbers, cf. 7.121.
(3.16) $u_0 | \theta^r | \sim u_0\mu\alpha_1^{(2r)} \sim \alpha_1^{(2r)}$. (3.17) $\varphi\alpha_1^{(2r)} \sim \mu u_1\alpha_1^{(2r)} \sim u_1\theta^r$.
(3.18) $| \theta^r | \nu \sim \mu\alpha_1^{(2^r)}u_0^2 \sim u_0^2 | \mu^2(2^r-1)^\nu | \sim \alpha_2^{(2^{r-1})}$.

(3.19) $\alpha_1^{(2^r)}\theta \sim \mu u_0\alpha_1^{(2^r)} \mid \mu^2 \mid u_0 \sim \mu\alpha_1^{(2^r)} \mid \mu^2 \mid u_0^2 \sim \mid \theta^r \mid \alpha_1^{(2)}$. For 3.20; cf. 7.121.
(4.1) $\zeta_0 \sim u_0^2 \sim \nu$; and $u_r\zeta_{-r} \simeq \zeta_r$. (4.2) $\zeta_r\varphi \sim u_0u_r\mu u_1 \sim u_ru_1 \sim \mid u_1\zeta_{r-1} \mid$.
(4.3) $u_r\zeta_s \sim u_ru_0u_s \sim u_s\zeta_r$. (4.4) $\theta\zeta_r \sim \mid \mu^2 \mid u_0^2u_r \sim u_r\alpha_1^{(2)}$.
(4.5) $\mid \varpi\alpha_1^{(2)} \mid \zeta_r \sim \mu\varpi^2u_0u_r \sim \varpi^2u_r \sim u_r \mid \nu\varpi \mid$. (4.6) $u_r\zeta_r \sim u_0u_r^2 \sim u_0 \mid u_r\nu \mid$.
(4.7) $u_0 \mid u_r\zeta_r \mid \sim u_0u_r^2 \mid \varpi u_r \mid \sim \mid \varpi u_r \mid u_ru_ru_0 \sim u_{2r}\zeta_r$.
(4.8) $u_r\zeta_{3r} \sim u_0u_ru_{3r} \sim u_0 \mid u_r\zeta_{2r} \mid$.
(4.9) $\mid u_{m+n}\zeta_{l-n} \mid \zeta_n \sim u_{m+l}u_{m+n}u_0u_n \sim \mid u_n\zeta_m \mid \zeta_{l+m}$.
(4.10) $\mid u_m\zeta_{n+l} \mid \zeta_n \sim u_{m+n+l}u_mu_0u_n \sim \mid u_n\zeta_{m+l} \mid \zeta_m$.
(4.11) In (4.10) put $l = 0$; $\mid u_m\zeta_n \mid \zeta_n \sim \mid u_n\zeta_m \mid \zeta_m$. (4.12) $u_0\varpi\zeta_m \sim k_2\zeta_m$.
(4.13) $\mid \theta^n \mid \zeta_m \sim u_m\alpha_1^{(2^n)}$; and (4.14) $\alpha_1^{(2m)}\zeta_n \sim \mid \nu\alpha_1^{(m)} \mid u_n$, from substituting

the generators, or from the resolutions into prime factors, as in the others.

7.11. The composites, [3.11], $u_0\psi$, $\mu\psi$ have been called by some [e. g., Bugajieff; Cahen; De Seguier], respectively *numerical integral*, $\int\psi$, and *numerical derivative*, $D\psi$, of ψ. It has not been considered necessary to retain this special terminology for these simplest composites, whose fundamental property is; if $\psi_1 \sim \int\psi \simeq u_0\psi$, then $\psi \sim D\psi_1 \simeq \mu\psi_1$, which is obvious since $u_0\mu$ is a unit.[1] Occasionally, $\psi_1\psi_2$ has been denoted by $\int\int\psi_1 \times \psi_2$; but the notations and ideas do not seem to have been generalized; all are the very simplest composites, and all their properties are evident from 6.26; also, the analogy with the integral calculus is very slight, and the term numerical integral, is reserved in the present treatment for a function which has close geometrical analogies with the properties of integrals. The following is the simplest illustration of a process treated generally farther on, viz., the derivation of new theorems by transformation of the generator. The general process enables any function to be expressed as a mixed sum of chosen functions.

†7.12. If, and only if, n is a perfect rth power, $\psi(\sqrt[r]{n})$ exists, $\psi(\ )$ being a numerical function. Writing $\psi_{1/r}(n) \equiv \psi(\sqrt[r]{n})$, and $\mid k_r\psi_{1/r} \mid \equiv [\psi]_r$; clearly, $[\psi]_r$, for the argument n, has the value 0 or $\psi_{1/r}(n)$ according as n is not or is a perfect rth power; also, if $\gamma(\psi) \equiv f(p, z)$, then $\gamma([\psi]_r) = f(p, z^r)$. That is, the generating function, $\gamma(\psi)$, of ψ, becomes, by the change of z into z^r, the generating function, $\gamma([\psi]_r)$, of $[\psi]_r$; or, if $[f(p, z)] = \sum_{n=1}^{\infty}\psi(n)/n^s$; then $[f(p, z^r)] = \sum_{n=1}^{\infty}\psi(n)/n^{rs} = \sum_{n=1}^{\infty}\psi'(n)/n^s$ where $\psi' \simeq [\psi]_r$. Again, the (ideal) product $\psi_1[\psi_2]_r$, evidently takes for the argument n, the simpler form $\Sigma'\psi_2(D)\psi_1(n/D^r)$, where Σ' *refers to all those divisors D of n which are such that n/D^r is an integer.* Similarly, it is easily seen that the simultaneous change of z into z^r and p into p^s in $f(p, z)$, giving $f(p^s, z^r)$, gives rise to ideal products, which, for the argument n are of the form $\Sigma'\psi_2(D^s)\psi_1(n^s/D^r)$; Σ' having the same significance as above. Performing these changes in the generators, reducing as in 7.01 the results into irreducible (algebraic) factors, new theorems

[1] For the corresponding product theorems of Dedekind and others, cf. § 8.

connecting Σ'-functions with Σ-functions (or ideal products), arise. All Σ''s are ideal products, or Σ-functions, of particular kinds, but, in Σ' or $[\ ]_r$ notation, new and interesting properties of the functions in relation to their arguments the integers, are revealed. Also, as in the preceding parts of this paper, relations solely between Σ''s are derived if desired. The following examples, among the simplest of these kinds, will suffice as illustrations; the first (3.09) is given in detail and the rest follow similarly from 7.12 and theorems in 7.10. Unless otherwise stated, D is any square divisor of n, and Σ' refers to all such D's.

†7.121. (3.9) $\Sigma'\varphi(D)\nu(n/D^2) = \Sigma' D\theta(n/D^2)$. For, $\gamma(\varphi) = (1-z)/(1-pz)$; $\gamma(\nu) = 1/(1-z)^2$; hence, [by 7.12], the $\gamma(\ )$ of the left is $(1-z^2)/(1-pz^2) \times 1/(1-z)^2$; but this is $1/(1-pz^2) \times (1+z)/(1-z)$; and $\gamma(u_1) = 1/(1-pz)$; $\gamma(\theta) = (1+z)/(1-z)$, hence, applying 7.12 again, the generator of the right is $1/(1-pz^2) \times (1+z)/(1-z)$; hence the theorem. In precisely the same way:

(3.10) $\Sigma'\theta(D)\nu(n/D^2) = \Sigma'\nu(D^2)\theta(n/D^2)$; remarking that $\theta(D) \equiv \theta(D^2)$, the left is generated by $(1+z^2)/(1-z^2)\times 1/(1-z)^2$, and the right by $(1+z^2)/(1-z^2) \times (1+z)/(1-z)$; these are identical, hence, etc.; and

(3.11) $\Sigma'\theta(D^2)\nu(n/D^2) = \Sigma'\nu(D^2)\theta(n/D^2)$; since $\theta(D) \equiv \theta(D^2)$.

(3.12) $\Sigma'\nu(D)\nu(D^r)\theta(n/D^2) = \Sigma'\nu(D^{2r})\nu(n/D^2)$. The respective generators are taken from 7.07 (9), (4) and (8), (1); in the first of each pair, z is replaced by z^2; giving, as the generators respectively of the left and right, $(1+(2r-1)z^2)/(1-z^2)^3 \times (1+z)/(1-z)$, and $(1+(2r-1)z^2)/(1-z^2)^2\times 1(1-z)^2$; these being identical, the theorem follows.

(3.13) $\Sigma\varpi(n/d)\nu(d)\nu(d^r) = \Sigma'\nu(n^{2r}/D^{4r})$; the summation on the left extending over all divisors d of n. As in (3.12), the respective generators are $1/(1+z) \times (1+(2r-1)z)/(1-z)^3$ and $1/(1-z^2) \times (1+(2r-1)z)/(1-z)^2$; hence, etc.

(3.14) $\Sigma\varpi(d)\sigma(d) = n\varpi(n)\Sigma' 1/D^2$. [Note that $\varpi(n/D^2) \equiv \varpi(n)$.]

Generator of left is $1/(1-z^2)(1+pz)$; of right, similarly to any of above, it is $1(1+pz) \times 1/(1-z^2)$; for, obviously, $\Sigma n/D^2\cdot\varpi(n/D^2) = n\varpi(n)\Sigma 1/D^2$; hence the theorem.

(3.15) Σ'' refers to all fourth-power divisors D^4, of n; $\Sigma'\varpi(D)\nu(n/D^2) = \Sigma''\theta(n/D^4)$. Similarly to the foregoing, remarking that

$$\sum_{n=1}^{\infty} 1/n^{4s} \times \sum_{n=1}^{\infty} \theta(n)/n^s = \sum_{n=1}^{\infty} \theta(n/D^4)/n^s,$$

and supplying the generators from 7.07.

(3.20) $\Sigma\nu(d^{2r})\varpi(n/d) = \Sigma'[\varpi(n/D^2)]^r$.

For, $(1+(2^r-1)z)/(1-z)^2 \times 1/(1+z) \equiv (1+(2^r-1)z)/(1-z)\times 1/(1-z^2)$.

(4.13) $\Sigma\varpi(d)\zeta_r(n/d) = \Sigma'(n/D^2)^r$.

For, $1/(1+z) \times 1/(1-z)(1-p^rz) \equiv 1/(1-z^2) \times 1/(1-p^rz)$.

7.13. All of these may be proved more briefly as in 7.10; but they are of sufficient interest to deserve writing out in full. In the same way, each

result in 7.10 may be revised to give theorems regarding square or rth power divisors only. All these processes are generalized later.

Note that $k_r u_0$, for the argument n, is the *number* of divisors of n which are perfect rth powers. Hence (see 7.10 (2.8)), $\Sigma\varpi(d) = 1$ or 0 according as n is or is not a perfect square; and (7.10 (2.14)) $\Sigma\varpi(d)\nu(n/d)$ is the number of square divisors of n. Similarly, on observing the *forms* of the ideal products, and translating the symbols into their arithmetical definitions, an endless variety of similar results relative to integers may be read off with great ease from the special theorems of § 7. Obviously, there is sufficient material in 7.05; 7.07; 7.08 for an inexhaustible fund of such results, and all these are but simple deductions from the single function Ψ $(n; a, b, c, l)$. Theorems relative to ideal addition do not present themselves until the extent[1] of the functions is $\geqq 3$. Further special theorems are given in § 8, and more, subsequently; those in § 8 are of a rather more general character than those in § 7, and present some aspects of these functions that do not seem to have been previously considered.

[1] So far as is known to the writer, no factorable numerical function which is a prime primitive, of extent $\geqq 3$, has as yet been considered in arithmetic, and there is but a single prime primitive of extent = 2 in use, due to G. Cantor; cf. P. Bachmann: *Die Analytische Zahlentheorie* (Leizpig, 1894), p. 327, Eqq. 43, 44, 45, wherein α, γ each contains such a factor, when latent units are expressed.